Symmetry breaking on graphs and groups

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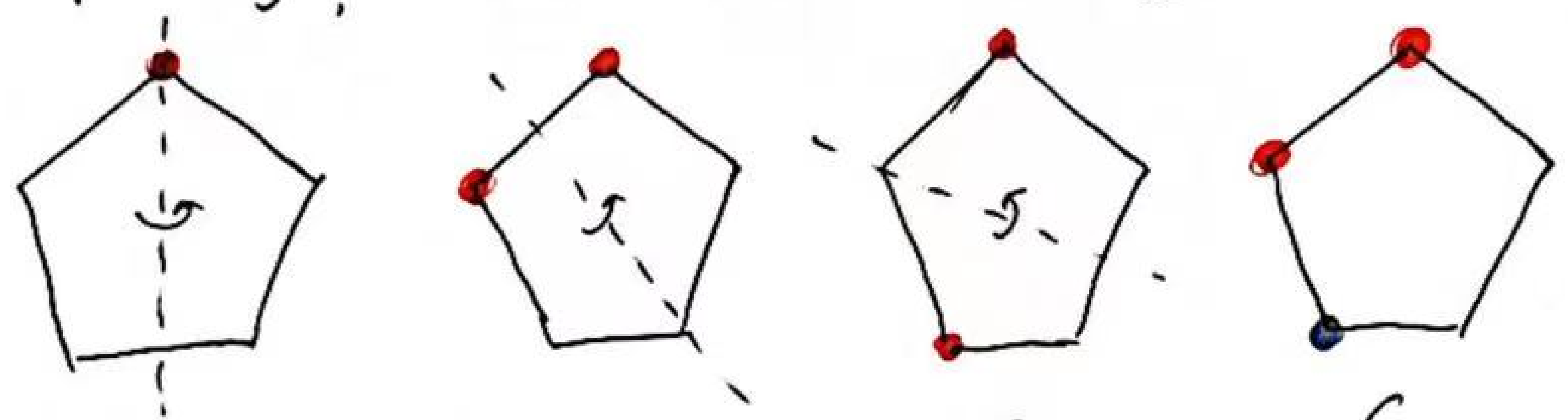
1. Colourings

Consider $\Gamma = C_5$, where $\mathrm{Aut}(\Gamma) \cong D_{10}$



C_A $\qquad$ C_B $\qquad$ C_C $\qquad$ C_D

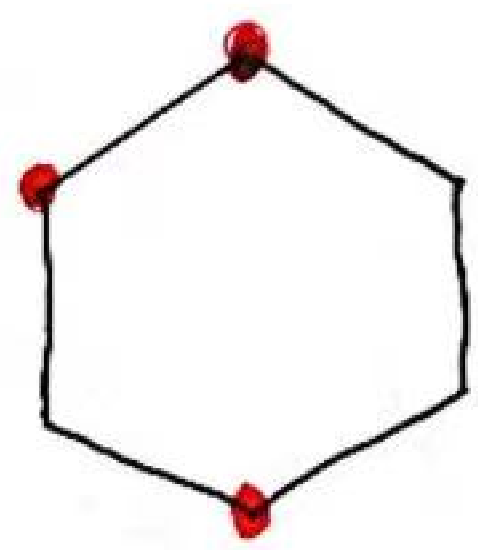
- $\mathrm{Aut}(\Gamma, C_A) \cong \mathrm{Aut}(\Gamma, C_B) \cong \mathrm{Aut}(\Gamma, C_C) \cong \mathbb{Z}_2$
- $\mathrm{Aut}(\Gamma, C_D) = 1$.

Distinguishing colouring: A colouring C of Γ s.t. $\mathrm{Aut}(\Gamma, C) = 1$.

Distinguishing number $D(\Gamma)$: The min number of colours in a distinguishing colouring of Γ.

Examples

- $D(C_5) = 3$
- $D(C_n) = 2$ for $n \geq 6$



- $D(K_n) = n$

Let $G \leq \mathrm{Sym}(\Omega)$ be a <u>transitive</u> permutation group. $|\Omega| = n$.

<u>Distinguishing partition</u> : A partition $\Pi = \{\pi_1, \dots, \pi_m\}$ s.t.

$$\bigcap_{i=1}^{m} G_{\{\pi_i\}} = 1.$$

<u>Distinguishing number $D(G)$</u> : The min size of a dist. partition.

- $D(\Gamma) = D(\mathrm{Aut}(\Gamma))$

<u>Examples</u>

- $D(D_{10}) = 3$
- $D(D_{2n}) = 2$ for $n \geq 6$
- $D(S_n) = n$
- $D(G) = 1 \iff G = 1$
- $G \neq 1$ is <u>regular</u> $\Rightarrow D(G) = 2$

<u>Note</u> $D(G) \leq 2 \iff \exists\, \Delta \subseteq \Omega$ s.t. $G_{\{\Delta\}} = 1$.

<u>Recall</u> G is called <u>primitive</u> if $G_\alpha <_{\max} G$.

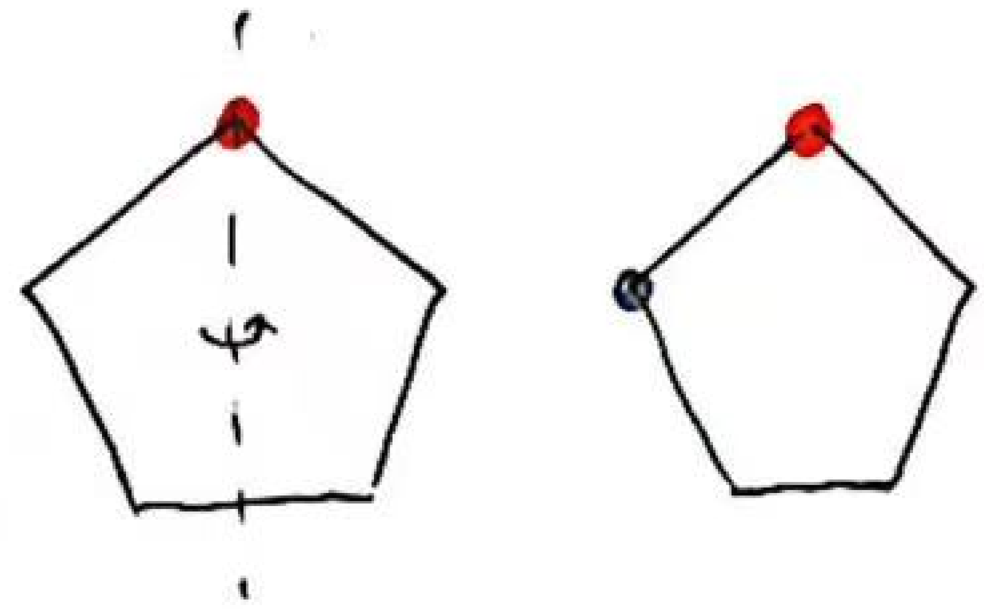
<u>Thm</u> (Cameron, Neumann & Saxl, 1984 ; Seress, 1997)

$G \notin \{A_n, S_n\}$ primitive $\Rightarrow D(G) = 2$, with 43 exceptions of degree ≤ 32 (e.g. D_{10}).

Let $G = \mathrm{Hol}(T) = T : \mathrm{Aut}(T) \leq \mathrm{Sym}(T)$, where T is a non-abelian simple group.

<u>Thm</u> (H, 2023+) $\forall\, 3 \leq k \leq |T| - 3$, $\exists\, \Delta \subseteq T$ s.t. $|\Delta| = k$ and $G_{\{\Delta\}} = 1$.

2. Fixing sets



<u>Fixing set</u> : A subset $\Delta \subseteq V\Gamma$ s.t. $\bigcap_{\alpha \in \Delta} \mathrm{Aut}(\Gamma)_\alpha = 1$

<u>Fixing number $\mathrm{fix}(\Gamma)$</u> : The min size of a fixing set.

<u>Examples</u>

- $\mathrm{fix}(C_n) = 2$ for $n \geq 3$
- $\mathrm{fix}(K_n) = n-1$

<u>Note</u> $D(\Gamma) \leq \mathrm{fix}(\Gamma) + 1$

Consider $G \leq \mathrm{Sym}(\Omega)$, where $|\Omega| = n$.

<u>Base</u> A subset $\Delta \subseteq \Omega$ s.t. $\bigcap_{\alpha \in \Delta} G_\alpha = 1$.

<u>Base size $b(G)$</u> : The min size of a base for G.

- $\mathrm{fix}(\Gamma) = b(\mathrm{Aut}(\Gamma))$
- $D(G) \leq b(G) + 1$

<u>Examples</u>

- $b(D_{2n}) = 2$ for $n \geq 3$
- $b(S_n) = n-1$
- $G = GL_d(q)$, $\Omega = \mathbb{F}_q^d \setminus \{0\} \Rightarrow b(G) = d$.

  Klavžar, Wong & Zhu, 2006 : $D(G) = 2$ if $\mathbb{F}_q^d \neq \mathbb{F}_2^2, \mathbb{F}_2^3, \mathbb{F}_4^2, \mathbb{F}_3^2$

- $G = S_m$, $\Omega = \{k\text{-subsets of } [m]\}$, $2k \le m$.

$b(G) =$ smallest ℓ s.t.

$$\sum_{\substack{\pi \vdash m \\ \pi = (1^{c_1}, \dots, m^{c_m})}} (-1)^{m - \sum c_i} \frac{m!}{\prod i^{c_i} c_i!} \left(\sum_{\substack{\eta \vdash k \\ \eta = (1^{b_1}, \dots, k^{b_k})}} \prod \binom{c_j}{b_j} \right)^{\ell} \neq 0$$

by Meceners & Spiga, 04/08/23

same (?) result by del Valle & Roney-Dougal, 08/08/23.

<u>Note</u> If Δ is a base and $x, y \in G$, then

$$\alpha^x = \alpha^y \ \forall \alpha \in \Delta \iff x^{-1} y \in \bigcap_{\alpha \in \Delta} G_\alpha$$
$$\iff x = y$$

That is,

elements of $G \xleftrightarrow{1-1}$ images of Δ.

<u>O'Nan - Scott theorem</u>

Finite primitive groups are divided into 5 types:

- Affine
- Almost simple
- Diagonal type
- Product type
- Twisted wreath product

3. Diagonal type

Let T be a non-abelian finite simple group and let

$$X = \{ (x, \dots, x) : x \in T \} \leq T^k.$$

Then $T^k \leq \mathrm{Sym}(\Omega)$, where $\Omega = [T^k : X]$.

A group G is said to be diagonal type if

$$T^k \trianglelefteq G \leq N_{\mathrm{Sym}(\Omega)}(T^k) \cong T^k.(\mathrm{Out}(T) \times S_k)$$

Note G induces $P_G \leq S_k$

Lemma G is primitive $\iff$ P_G is primitive, or $\boxed{k=2 \ \& \ P_G=1}$

$$T : \mathrm{Inn}(T) \trianglelefteq G \leq T : \mathrm{Aut}(T) = \mathrm{Hol}(T)$$

Thm (Fawcett, 2013) $P_G \notin \{A_k, S_k\} \Rightarrow b(G) = 2$.

key observation $b(G) = 2$ if

$$\exists\, \Delta \subseteq T \text{ s.t. } |\Delta| = k \text{ and } \mathrm{Hol}(T)_{\{\Delta\}} = 1 \quad (*)$$

Recall $3 \leq k \leq |T| - 3 \Rightarrow (*) \Rightarrow b(G) = 2$.

Thm (H, 2023+) $b(G) = 2 \iff$ one of the following:

(i) $P_G \notin \{A_k, S_k\}$

(ii) $3 \leq k \leq |T| - 3$

(iii) $k \in \{|T| - 2, |T| - 1\}$ and $S_k \ntrianglelefteq G$.

Thm (H, 2023+) Base sizes of diagonal type primitive groups are determined.

4. Probabilistic method

Let $\mathbb{P}(G)$ be the probability that a random element of X satisfies a property E.

e.g.

X	E	if $\mathbb{P}(G) > 0$
$\{\text{subsets } \Delta \subseteq \Omega\}$	"$G_{\{\Delta\}} = 1$"	$D(G) = 2$
$\{k\text{-subsets } \Delta \subseteq \Omega\}$	"$G_{\{\Delta\}} = 1$"	$(*)$
$\{(\alpha_1, \dots, \alpha_k) \in \Omega^k\}$	"$\bigcap_{i=1}^{k} G_{\alpha_i} = 1$"	$b(G) \leq k$

Let $\mathbb{Q}(G) := 1 - \mathbb{P}(G)$ and suppose $\widehat{\mathbb{Q}}(G) > \mathbb{Q}(G)$.

<u>Note</u> $\widehat{\mathbb{Q}}(G) < 1 \Rightarrow \exists$ an element of X satisfying E.

<u>Example</u> $X = \Omega^k$, $E =$ "$\bigcap_{i=1}^{k} G_{\alpha_i} = 1$". Then

$$\mathbb{Q}(G) < \sum_{\substack{x \in G \\ |x| \text{ prime}}} \mathrm{fpr}(x)^k = \sum_{\substack{x \in G \\ |x| \text{ prime}}} \left(\frac{|x^G \cap G_\alpha|}{|x^G|} \right)^k =: \widehat{\mathbb{Q}}(G)$$