Bases for permutation groups

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1. Bases

Let $G \leq \mathrm{Sym}(\Omega)$, where $|\Omega| < \infty$ and G is <u>transitive</u>.

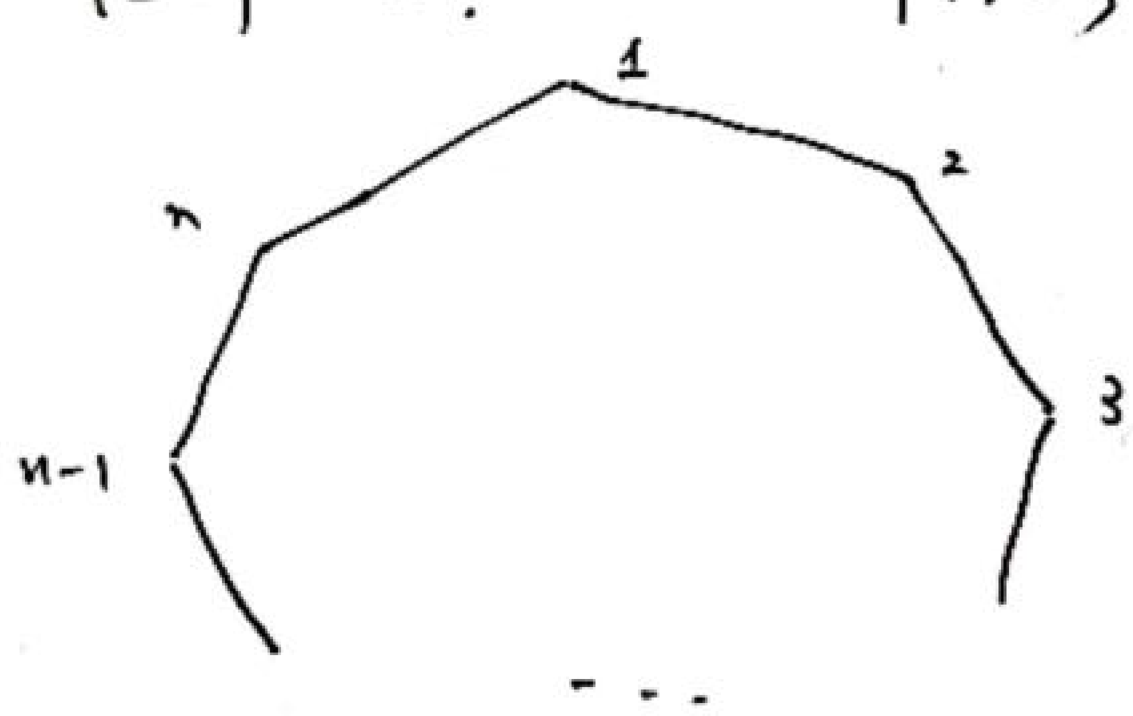
Point stabiliser: $G_\alpha = \{ g \in G : \alpha^g = \alpha \}$.

<u>Note</u>. $\bigcap_{\alpha \in \Omega} G_\alpha = 1$.

<u>Question</u>. Any subset $\Delta \subseteq \Omega$ with $\bigcap_{\alpha \in \Delta} G_\alpha = 1$?

<u>Examples</u>.

- $G = S_n$, $|\Omega| = n$, $\Delta = \{1, \dots, n-1\}$ $\boxed{b(G) = n-1}$ $\boxed{r(G) = 1}$
- $G = D_{2n}$, $|\Omega| = n$. $\Delta = \{1, 2\}$ $\boxed{b(G) = 2}$ $\boxed{r(G) = \lceil \frac{n}{2} \rceil - 1}$



- $G = GL(V)$, $\Omega = V \setminus \{0\}$

  Δ contains a basis of V. $\boxed{b(G) = \dim V}$ $\boxed{r(G) = 1}$
- $G = S_n$. $\Omega = \{k\text{-subsets of } [n]\}$, $2k \leq n$

  $\Delta = \{ \{1, \dots, k\}, \{2, \dots, k+1\}, \cdots \{n-k+1, \dots, n\} \}$

$b(G) = ?$ Very complicated!
Mecenero & Spiga, 04/08/23
del Valle & Roney-Dougal, 08/08/23

$\boxed{r(G) = ?}$

Def. • $\Delta \subseteq \Omega$ is called a <u>base</u> for G if $\bigcap_{\alpha \in \Delta} G_\alpha = 1$.

• The <u>base size</u> of G, denoted $b(G)$, is the minimal size of a base for G.

<u>Note</u> $b(G) = \min\{k \mid G \text{ has a regular orbit on } \Omega^k\}$.

<u>Q1</u> Determine $b(G)$?

<u>Q2</u> Classify G with $b(G) = 2$?

Let $r(G)$ be the number of regular G-orbits on $\Omega^{b(G)}$.

<u>Q3</u> Determine $r(G)$?

<u>Q4</u> Classify G with $r(G) = 1$?

<u>Lower bound</u>

Let Δ be a base of size $b(G)$ and $x, y \in G$. Then

$$\alpha^x = \alpha^y \ \forall \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha$$
$$\iff x = y.$$

That is,

$$\text{elements of } G \xleftrightarrow{1-1} \text{images of } \Delta.$$

Hence, $|G| \leq |\Omega|^{b(G)}$ and so $b(G) \geq \log_{|\Omega|}|G|$.

<u>Upper bound</u> $b(G) \leq \log_2 |G|$

<u>Primitive groups</u>

"Primitive" = "transitive" + "$G_\alpha \underset{\max}{<} G$".

e.g. $G = D_{2n}$, $|\Omega| = n$. Then G is primitive $\iff$ n is prime.

Halasi, Liebeck & Maróti, 2019: $b(G) \leq 2\log_{|\Omega|}|G| + 24$.

(originally Pyber's conjecture).

O'Nan - Scott

Finite primitive groups are divided into 5 types:

- Affine
- Almost simple
- Diagonal type
- Product type
- Twisted wreath product

2. Diagonal type

Let T be a non-abelian finite simple group and let

$$D = \{ (t, \dots, t) : t \in T \} \leq T^k$$

Then $T^k \leq Sym(\Omega)$ with $\Omega = [T^k : D]$

A group G is said to be of <u>diagonal type</u> if

$$T^k \trianglelefteq G \leq N_{Sym(\Omega)}(T^k) \cong T^k . (Out(T) \times S_k)$$

<u>Note</u> G induces $P_G \leq S_k$, so $T^k \trianglelefteq G \leq T^k . (Out(T) \times P_G)$

<u>Lemma</u> G is primitive $\iff$ P_G is primitive, or $\boxed{k = 2 \text{ and } P_G = 1}$

$$T : Inn(T) \trianglelefteq G \leq T : Aut(T) = Hol(T)$$

<u>Theorem</u> (Fawcett, 2013)

- $P_G \notin \{A_k, S_k\} \Rightarrow b(G) = 2$
- $P_G \in \{A_k, S_k\}$ and $b(G) = 2 \Rightarrow 2 < k < |T|$

<u>Key observation</u>

$b(G) = 2$ if $\exists S \subseteq T$ s.t. $|S| = k$ and $Hol(T)_{\{S\}} = 1$.

setwise stabiliser

An approach

Let $\mathcal{A} = \{ S \subseteq T : |S| = k \text{ and } \mathrm{Hol}(T)_{\{S\}} \neq 1 \}$.

Suppose $S \in \mathcal{A}$. Then $\exists\, \sigma \in \mathrm{Hol}(T)_{\{S\}}$ of prime order.

Thus,
$$S \in \mathrm{fix}(\sigma, k) = \{ S \subseteq T : |S| = k \text{ and } \sigma \in \mathrm{Hol}(T)_{\{S\}} \}.$$

Let $\mathcal{P}$ be the set of elements of $\mathrm{Hol}(T)$ of prime order.

Then
$$|\mathcal{A}| = \Big| \bigcup_{\sigma \in \mathcal{P}} \mathrm{fix}(\sigma, k) \Big|$$
$$\leq \sum_{\sigma \in \mathcal{P}} |\mathrm{fix}(\sigma, k)| =: m$$

<u>Note</u> $b(G) = 2$ if $m < \binom{|T|}{k}$.

3. Results (Here G is a diagonal type primitive group).

<u>Theorem</u> (H, 2024) If $3 \leq k \leq |T| - 3$, then $\exists\, S \subseteq T$ s.t.

$$|S| = k \quad \text{and} \quad \mathrm{Hol}(T)_{\{S\}} = 1.$$

<u>Theorem</u> (H, 2024) $b(G) = 2$ iff

- $P_G \notin \{A_k, S_k\}$
- $3 \leq k \leq |T| - 3$
- $k \in \{|T| - 2, |T| - 1\}$ and $S_k \not\leq G$.

<u>Theorem</u> (H, 2024) If $b(G) = 2$, then $r(G) = 1$ iff

$$G = T^k . (\mathrm{Out}(T) \times S_k), \quad T = A_5, \quad k \in \{3, 57\}$$

<u>Theorem</u> (H, 2024) $b(G)$ is computed in all cases.

<u>Theorem</u> (Freedman, H, Lee, & Rekvényi, 2024+) $b(G) > 2 \Rightarrow r(G) > 1$.