Symmetry breaking on primitive groups

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Seminars on Groups and Graphs

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University of BRISTOL

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How can we "break" the symmetries of a graph?

- Colouring vertices (setwise)
- Fixing vertices (pointwise)

Part I

Distinguishing numbers for groups and graphs

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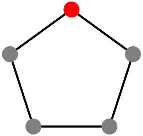
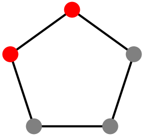
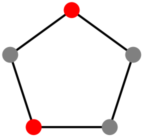
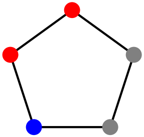
$C_1 =$ 

$C_2 =$ 

$C_3 =$ 

$C_4 =$ 

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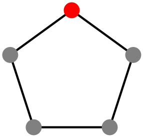
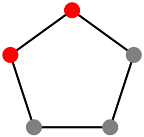
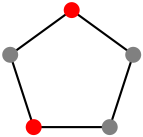
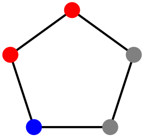
$C_1 =$ 

$C_2 =$ 

$C_3 =$ 

$C_4 =$ 

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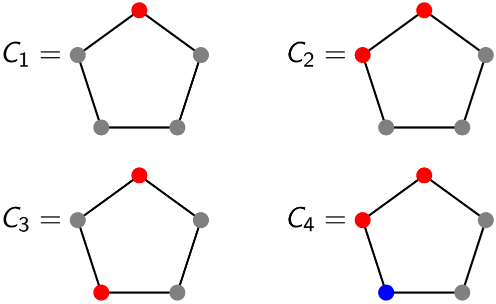

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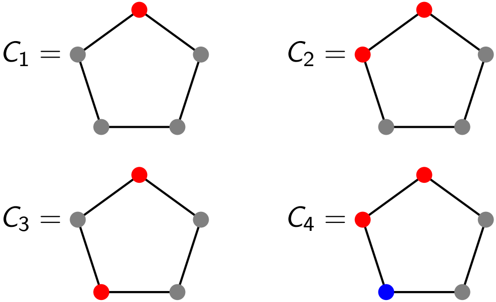

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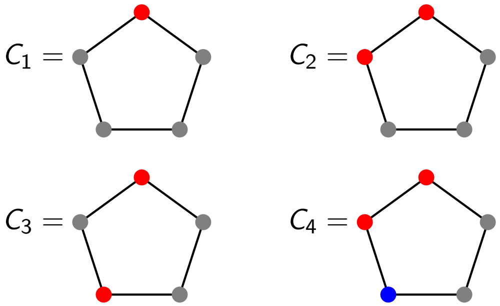

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- $G \neq 1$ is regular $\implies D(G) = 2$.

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Theorem (H, 2023+)

Suppose T non-abelian simple, $G = \mathrm{Hol}(T) = T{:}\mathrm{Aut}(T)$, $\Omega = T$ and $3 \leqslant k \leqslant |T| - 3$. Then

$$\exists\ \Delta \subseteq \Omega \text{ such that } |\Delta| = k \text{ and } G_{\{\Delta\}} = 1.$$

Part II

Bases for permutation groups

Fixing sets

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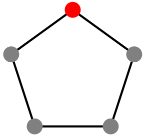

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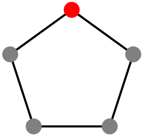

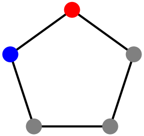

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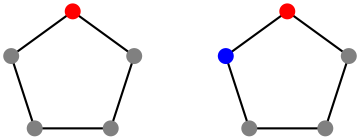

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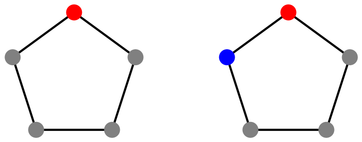

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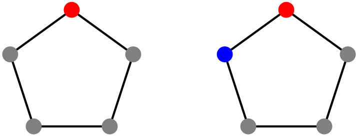

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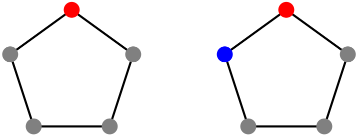

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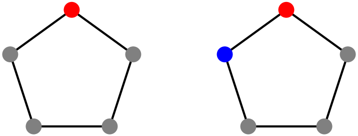

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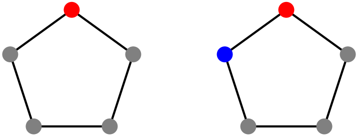

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- $D(\Gamma) \leqslant \mathrm{fix}(\Gamma) + 1$.

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  Klavžar, Wong & Zhu, 2006: $D(G) = 2$ if $\mathbb{F}_q^d \neq \mathbb{F}_2^2, \mathbb{F}_2^3, \mathbb{F}_4^2$ or $\mathbb{F}_3^2$.

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Thus, $2^{b(G)} \leqslant |G|$ and so $\log_{|\Omega|} |G| \leqslant b(G) \leqslant \log_2 |G|$.

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Soluble groups:

- **Seress, 1996:** G soluble $\implies b(G) \leqslant 4$
- **Burness, 2021:** G_α soluble $\implies b(G) \leqslant 5$

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- **Burness & H, 2023:** partial results on $G < L \wr P$.

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- **Burness & Thomas, 2023:** $b(G) = 2$ if G is exceptional and G_α is the normaliser of a maximal torus ✓

Part III

Base sizes of diagonal type primitive groups

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Thus, $T^k \trianglelefteq G \leqslant T^k.(\mathrm{Out}(T) \times P)$.

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Set $\alpha = D$ and suppose $G = T^k.(\mathrm{Out}(T) \times P)$. Then

$$G_\alpha = \{(a, \ldots, a)\pi : a \in \mathrm{Aut}(T), \pi \in P\} \cong \mathrm{Aut}(T) \times P.$$

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Remark. This method is not useful for $P \in \{A_k, S_k\}$.

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Note. If $g = (a, \dots, a)\pi \in G_\alpha \cap G_\beta$, then $t_{i^\pi} = x t_i^a$ for some $x \in T$, so $x^{-1}a \in X_{\{\Delta\}}$. Thus, $a = 1$ and $x = 1$, which implies $\pi = 1$ as $t_1, \dots, t_k$ are distinct. Hence $g = 1$.

Holomorph

Let $X = \mathrm{Hol}(T) = T{:}\mathrm{Aut}(T) \leqslant \mathrm{Sym}(T)$. Recall that

Theorem (H, 2023+)

If $3 \leqslant k \leqslant |T| - 3$, then $\exists\, \Delta \subseteq T$ such that $|\Delta| = k$ and $X_{\{\Delta\}} = 1$.

Assume $3 \leqslant k \leqslant |T| - 3$ and $G = T^k.(\mathrm{Out}(T) \times S_k)$.

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Theorem (H, 2023+)

If $3 \leqslant k \leqslant |T| - 3$ and $G = T^k.(\mathrm{Out}(T) \times S_k)$, then $b(G) = 2$.

Base sizes

Theorem (H, 2023+)

Suppose $G \leqslant T^k.(\mathrm{Out}(T) \times P)$ is a diagonal type primitive group of top group P. Then $b(G) = 2$ iff one of the following holds:

- $P \notin \{A_k, S_k\}$;
- $3 \leqslant k \leqslant |T| - 3$;
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Theorem (H, 2023+)

Suppose G is a diagonal type primitive group. Then $b(G)$ is known. ✓

Part IV

Connections and related problems

Subsets with trivial stabiliser

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a primitive group. Assume $G \notin \{\mathrm{Sym}(\Omega), \mathrm{Alt}(\Omega)\}$.

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Theorem (H, 2023+)

If G is **holomorph simple**, then G has the property $(\star)$.

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Let G be a non-abelian finite group. For $S \subseteq G \setminus \{1\}$, let

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Let T be a non-abelian finite simple group and define

$$\mathbb{P}_k(T) := \frac{|\{S \subseteq T \setminus \{1\} : |S| = k,\ \mathrm{Aut}(T,S) \neq 1\}|}{\binom{|T|-1}{k}}$$

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An application: existence of DRR with prescribed valency. (Pablo Spiga)

Bases and regular suborbits

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- **H, 2023+:** G diagonal type ✓
- **Burness & H, 2022:** G almost simple, G_α soluble ✓
- **Burness & H, 2023:** Some product type groups.

Thank you!