Bases for permutation groups

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Remark. The **determining number** of a graph Γ is $b(\mathrm{Aut}(\Gamma))$.

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- $b(G) = \dim V$.

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$b(G)$ = smallest integer ℓ such that

$$\sum_{\substack{\pi \,\vdash\, n \\ \pi=(1^{c_1},\dots,n^{c_n})}} (-1)^{n-\sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!} \left(\sum_{\substack{\eta \,\vdash\, k \\ \eta=(1^{b_1},\dots,k^{b_k})}} \prod_{j=1}^k \binom{c_j}{b_j} \right)^{\ell} \neq 0.$$

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In general, determining $b(G)$ is a very difficult problem!

Bounds

Let Δ be a base of size $b(G)$ and let $x, y \in G$. Then

$$\alpha^x = \alpha^y \text{ for any } \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha \iff x = y.$$

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Halasi, Liebeck & Maróti, 2019:

$$G \text{ primitive} \implies b(G) \leqslant 2 \log_{|\Omega|} |G| + 24.$$

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Product type: Partial results **(Burness & H, 2023)**

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Note. $b(G) = 2$ if $m < \binom{|T|}{k}$.

Main results

Theorem (H, 2023+)

If $3 \leqslant k \leqslant |T| - 3$, then $\exists\ S \subseteq T$ such that $|S| = k$ and $Y_{\{S\}} = 1$.

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If $3 \leqslant k \leqslant |T| - 3$, then $\exists\ S \subseteq T$ such that $|S| = k$ and $Y_{\{S\}} = 1$.

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Suppose $G \leqslant T^k.(\mathrm{Out}(T) \times P)$ is a diagonal type primitive group with top group P. Then $b(G) = 2$ if and only if one of the following holds:

- $P \notin \{A_k, S_k\}$;
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- $k \in \{|T| - 2, |T| - 1\}$ and $S_k \not\leqslant G$.

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Theorem (H, 2023+)

Suppose G is a diagonal type primitive group. Then $b(G)$ is computed. ✓

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Freedman, H, Lee & Rekvényi, 2023+:

$$G \text{ is a diagonal type primitive group with } b(G) > 2 \implies r(G) > 1.$$

Thank you!