The generalised Saxl graphs of finite permutation groups

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Topics in Group Theory

(on the occasion of Andrea Lucchini's 60th(+) birthday)

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Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a permutation group, with $|\Omega| < \infty$.

Base: a subset of Ω with trivial pointwise stabiliser.

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- $G = S_n$, $|\Omega| = n$:
  $\{1, \ldots, n-1\}$ is a base; $b(G) = n - 1$.
- $G = \mathrm{GL}(V)$, $\Omega = V \setminus \{0\}$:
  $\Delta \subseteq \Omega$ is a base $\iff$ Δ spans V; $b(G) = \dim V$.

Saxl graphs

Burness & Giudici (BG), 20: **Saxl graph** $\Sigma(G)$ when $b(G) = 2$:

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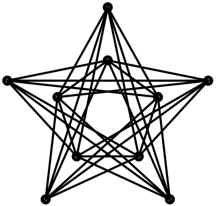
  For example, when $q = 4$ we have the complement of the Petersen.



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Freedman, H, Lee & Rekvényi (FHLR), 24+:

Generalised Saxl graph $\Sigma(G)$ $\boxed{\text{when } b(G) \geqslant 2}$:

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$x \sim y \iff \{x, y\}$ is a subset of a generating set of G of minimal size.

Basic properties

Throughout, let $\Sigma(G)$ be the **generalised** Saxl graph of G.

Example

Let $G = \mathrm{GL}(V)$ and $\Omega = V \setminus \{0\}$. Then $b(G) = \dim V$, and $\alpha \sim \beta$ iff α and β are linearly independent. So $\Sigma(G)$ is **complete multipartite**.

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- $G = D_8 \times D_8$ and $\Omega = \{1,2,3,4\}^2$: $\Sigma(G) = 2K_{4,4}$ (not connected).

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Affine & Product types: Partial results.

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- G almost simple sporadic and $b(G) \geqslant 3$ **(FHLR, 24+)**
- $G \leqslant T^k.(\mathrm{Out}(T) \times P)$ diagonal type, $P \notin \{A_k, S_k\}$ **(H, 24+)**
- $G = T^k{:}P$ twisted wreath product, P primitive **(H, 24+)**

Probabilistic methods

Let

$$Q(G,k) := \frac{|\{(\alpha_1,\dots,\alpha_k)\in\Omega^k : G_{\alpha_1}\cap\cdots\cap G_{\alpha_k}\neq 1\}|}{|\Omega|^k}$$

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Example

If $(G, G_\alpha) = (\mathrm{PGL}_2(q), D_{2(q-1)})$ then $\Sigma(G) = J(q+1,2)$ has the Common Neighbour Property, although $Q(G,b(G)) \to 1$ as $q \to \infty$.

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- G diagonal type ✓ **(H, 24; FHLR, 24+)**

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FHLR, 24+:

- A complete classification when $\mathrm{soc}(G) = \mathrm{PSL}_2(q)$ ✓
- Partial results when G is a sporadic group or of diagonal type.

Future work

- Study $\Sigma(G)$ when G is an imprimitive group.
- Study other graph invariants of $\Sigma(G)$ (e.g. clique number).
- ....

Thank you!