# Bases, distinguishing partitions and probabilistic methods

Hong Yi Huang

Discrete Structures and Algorithms Seminar

University of Melbourne

18 August 2022

University of BRISTOL

# Automorphisms

Throughout, everything is finite.

# Automorphisms

Throughout, everything is finite.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a simple graph.

# Automorphisms

Throughout, everything is finite.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a simple graph.

**Automorphism:** $g \in \mathrm{Sym}(V\Gamma)$ such that $v \sim w \iff v^g \sim w^g$.

# Automorphisms

Throughout, everything is finite.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a simple graph.

**Automorphism:** $g \in \mathrm{Sym}(V\Gamma)$ such that $v \sim w \iff v^g \sim w^g$.

**Automorphism group** $\mathrm{Aut}(\Gamma)$**:** The group of all the automorphisms.

# Automorphisms

Throughout, everything is finite.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a simple graph.

**Automorphism:** $g \in \mathrm{Sym}(V\Gamma)$ such that $v \sim w \iff v^g \sim w^g$.

**Automorphism group** $\mathrm{Aut}(\Gamma)$**:** The group of all the automorphisms.

- $\Gamma = \mathbf{K}_n \implies \mathrm{Aut}(\Gamma) \cong S_n$

# Automorphisms

Throughout, everything is finite.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a simple graph.

**Automorphism:** $g \in \mathrm{Sym}(V\Gamma)$ such that $v \sim w \iff v^g \sim w^g$.

**Automorphism group** $\mathrm{Aut}(\Gamma)$**:** The group of all the automorphisms.

- $\Gamma = \mathbf{K}_n \implies \mathrm{Aut}(\Gamma) \cong S_n$
- $\Gamma = \mathbf{C}_n \ (n \geqslant 3) \implies \mathrm{Aut}(\Gamma) \cong D_{2n}$

# Automorphisms

Throughout, everything is finite.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a simple graph.

**Automorphism:** $g \in \mathrm{Sym}(V\Gamma)$ such that $v \sim w \iff v^g \sim w^g$.

**Automorphism group** $\mathrm{Aut}(\Gamma)$**:** The group of all the automorphisms.

- $\Gamma = \mathbf{K}_n \implies \mathrm{Aut}(\Gamma) \cong S_n$
- $\Gamma = \mathbf{C}_n \ (n \geqslant 3) \implies \mathrm{Aut}(\Gamma) \cong D_{2n}$

How can we "break" the symmetries of a graph?

# Automorphisms

Throughout, everything is finite.

Let $\Gamma = (V\Gamma, E\Gamma)$ be a simple graph.

**Automorphism:** $g \in \mathrm{Sym}(V\Gamma)$ such that $v \sim w \iff v^g \sim w^g$.

**Automorphism group** $\mathrm{Aut}(\Gamma)$**:** The group of all the automorphisms.

- $\Gamma = \mathbf{K}_n \implies \mathrm{Aut}(\Gamma) \cong S_n$
- $\Gamma = \mathbf{C}_n \ (n \geqslant 3) \implies \mathrm{Aut}(\Gamma) \cong D_{2n}$

How can we "break" the symmetries of a graph?

- Colouring vertices (setwise)
- Fixing vertices (pointwise)

# Part I

Distinguishing numbers for groups and graphs

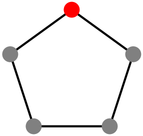
# Colourings

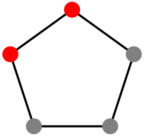
Consider $\Gamma = \mathbf{C}_5$, where we have $\mathrm{Aut}(\Gamma) \cong D_{10}$.

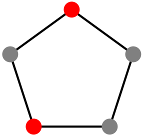
# Colourings

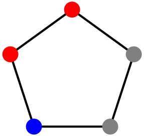
Consider $\Gamma = \mathbf{C}_5$, where we have $\mathrm{Aut}(\Gamma) \cong D_{10}$.

Which automorphisms preserve the following colourings?

$C_1 =$ 

$C_2 =$ 

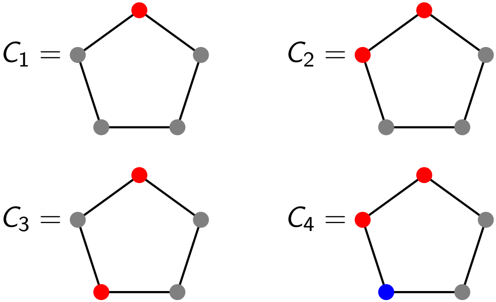
$C_3 =$ 

$C_4 =$ 

# Colourings

Consider $\Gamma = \mathbf{C}_5$, where we have $\mathrm{Aut}(\Gamma) \cong D_{10}$.

Which automorphisms preserve the following colourings?

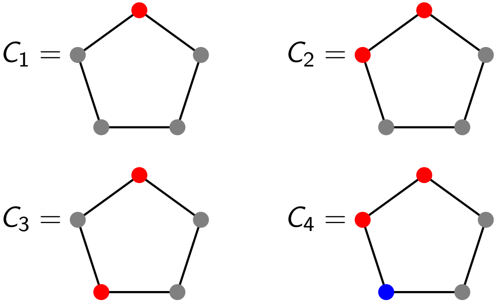



- $\mathrm{Aut}(\Gamma, C_1) \cong \mathrm{Aut}(\Gamma, C_2) \cong \mathrm{Aut}(\Gamma, C_3) \cong \mathbb{Z}_2$, and $\mathrm{Aut}(\Gamma, C_4) = 1$.

# Colourings

Consider $\Gamma = \mathbf{C}_5$, where we have $\mathrm{Aut}(\Gamma) \cong D_{10}$.

Which automorphisms preserve the following colourings?



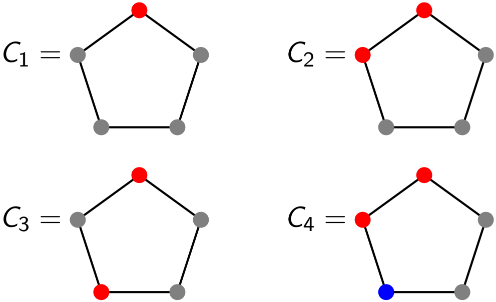

- $\mathrm{Aut}(\Gamma, C_1) \cong \mathrm{Aut}(\Gamma, C_2) \cong \mathrm{Aut}(\Gamma, C_3) \cong \mathbb{Z}_2$, and $\mathrm{Aut}(\Gamma, C_4) = 1$.

**Distinguishing colouring:** A colouring $C$ of $\Gamma$ such that $\mathrm{Aut}(\Gamma, C) = 1$.

# Colourings

Consider $\Gamma = \mathbf{C}_5$, where we have $\mathrm{Aut}(\Gamma) \cong D_{10}$.

Which automorphisms preserve the following colourings?




- $\mathrm{Aut}(\Gamma, C_1) \cong \mathrm{Aut}(\Gamma, C_2) \cong \mathrm{Aut}(\Gamma, C_3) \cong \mathbb{Z}_2$, and $\mathrm{Aut}(\Gamma, C_4) = 1$.

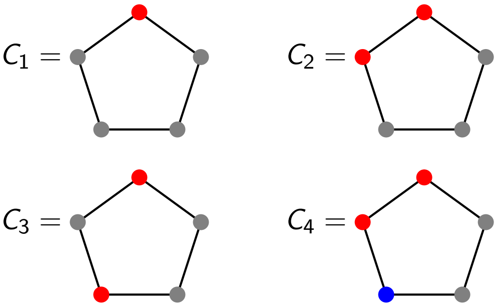
**Distinguishing colouring:** A colouring $C$ of $\Gamma$ such that $\mathrm{Aut}(\Gamma, C) = 1$.

**Distinguishing number $D(\Gamma)$:** The minimal number of colours in a distinguishing colouring of $\Gamma$.

# Colourings

Consider $\Gamma = \mathbf{C}_5$, where we have $\mathrm{Aut}(\Gamma) \cong D_{10}$.

Which automorphisms preserve the following colourings?




- $\mathrm{Aut}(\Gamma, C_1) \cong \mathrm{Aut}(\Gamma, C_2) \cong \mathrm{Aut}(\Gamma, C_3) \cong \mathbb{Z}_2$, and $\mathrm{Aut}(\Gamma, C_4) = 1$.

**Distinguishing colouring:** A colouring $C$ of $\Gamma$ such that $\mathrm{Aut}(\Gamma, C) = 1$.

**Distinguishing number $D(\Gamma)$:** The minimal number of colours in a distinguishing colouring of $\Gamma$. (e.g. $D(\mathbf{C}_5) = 3$.)

# Some results

**Klavžar, Wong & Zhu, 2006:**

$D(\Gamma) \leqslant$ the maximal degree of $\Gamma$, unless $\Gamma \cong \mathbf{K}_n$, $\mathbf{K}_{n,n}$ or $\mathbf{C}_5$.

# Some results

**Klavžar, Wong & Zhu, 2006:**

$D(\Gamma) \leqslant$ the maximal degree of $\Gamma$, unless $\Gamma \cong \mathbf{K}_n$, $\mathbf{K}_{n,n}$ or $\mathbf{C}_5$.

**Remark.** $D(\mathbf{K}_n) = n$; $D(\mathbf{K}_{n,n}) = n + 1$; $D(\mathbf{C}_n) = 2$ if $n \geqslant 6$.

# Some results

**Klavžar, Wong & Zhu, 2006:**

$D(\Gamma) \leqslant$ the maximal degree of $\Gamma$, unless $\Gamma \cong \mathbf{K}_n$, $\mathbf{K}_{n,n}$ or $\mathbf{C}_5$.

**Remark.** $D(\mathbf{K}_n) = n$; $D(\mathbf{K}_{n,n}) = n+1$; $D(\mathbf{C}_n) = 2$ if $n \geqslant 6$.

**Praeger, 1993; Devillers, Harper & Morgan, 2019:**

If $\Gamma$ is **2-arc-transitive**, then one of the following holds.

- $\Gamma$ is complete;
- $\Gamma$ is bipartite;
- $D(\Gamma) = 2$;
- $\Gamma \cong \mathbf{C}_5$, $\mathbf{K}_3 \square \mathbf{K}_3$, Petersen or its complement.

# Partitions

**Note.** A colouring is a partition of vertices.

# Partitions

**Note.** A colouring is a partition of vertices.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a **transitive** permutation group of degree $n$.

# Partitions

**Note.** A colouring is a partition of vertices.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a **transitive** permutation group of degree $n$.

**Distinguishing partition:** A partition $\Pi = \{\pi_1, \ldots, \pi_m\}$ of $\Omega$ such that

$$\bigcap_{i=1}^{m} G_{\{\pi_i\}} = 1.$$

# Partitions

**Note.** A colouring is a partition of vertices.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a **transitive** permutation group of degree $n$.

**Distinguishing partition:** A partition $\Pi = \{\pi_1, \ldots, \pi_m\}$ of $\Omega$ such that

$$\bigcap_{i=1}^{m} G_{\{\pi_i\}} = 1.$$

**Distinguishing number $D(G)$:** The minimal size of a dist. partition.

# Partitions

**Note.** A colouring is a partition of vertices.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a **transitive** permutation group of degree $n$.

**Distinguishing partition:** A partition $\Pi = \{\pi_1, \ldots, \pi_m\}$ of $\Omega$ such that

$$\bigcap_{i=1}^{m} G_{\{\pi_i\}} = 1.$$

**Distinguishing number $D(G)$:** The minimal size of a dist. partition.

**Remark.** $D(\Gamma) = D(\mathrm{Aut}(\Gamma))$, so $D(D_{10}) = 3$ and $D(D_{2n}) = 2$ for $n \geqslant 6$.

# Partitions

**Note.** A colouring is a partition of vertices.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a **transitive** permutation group of degree $n$.

**Distinguishing partition:** A partition $\Pi = \{\pi_1, \ldots, \pi_m\}$ of $\Omega$ such that

$$\bigcap_{i=1}^{m} G_{\{\pi_i\}} = 1.$$

**Distinguishing number $D(G)$:** The minimal size of a dist. partition.

**Remark.** $D(\Gamma) = D(\mathrm{Aut}(\Gamma))$, so $D(D_{10}) = 3$ and $D(D_{2n}) = 2$ for $n \geqslant 6$.

## Examples

- $D(S_n) = n$

# Partitions

**Note.** A colouring is a partition of vertices.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a **transitive** permutation group of degree $n$.

**Distinguishing partition:** A partition $\Pi = \{\pi_1, \ldots, \pi_m\}$ of $\Omega$ such that

$$\bigcap_{i=1}^{m} G_{\{\pi_i\}} = 1.$$

**Distinguishing number $D(G)$:** The minimal size of a dist. partition.

**Remark.** $D(\Gamma) = D(\mathrm{Aut}(\Gamma))$, so $D(D_{10}) = 3$ and $D(D_{2n}) = 2$ for $n \geqslant 6$.

## Examples

- $D(S_n) = n$; $D(A_n) = n - 1$.

# Partitions

**Note.** A colouring is a partition of vertices.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a **transitive** permutation group of degree $n$.

**Distinguishing partition:** A partition $\Pi = \{\pi_1, \ldots, \pi_m\}$ of $\Omega$ such that

$$\bigcap_{i=1}^{m} G_{\{\pi_i\}} = 1.$$

**Distinguishing number $D(G)$:** The minimal size of a dist. partition.

**Remark.** $D(\Gamma) = D(\mathrm{Aut}(\Gamma))$, so $D(D_{10}) = 3$ and $D(D_{2n}) = 2$ for $n \geqslant 6$.

## Examples

- $D(S_n) = n$; $D(A_n) = n - 1$.
- $D(G) = 1 \iff G = 1$.

# Partitions

**Note.** A colouring is a partition of vertices.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a **transitive** permutation group of degree $n$.

**Distinguishing partition:** A partition $\Pi = \{\pi_1, \ldots, \pi_m\}$ of $\Omega$ such that

$$\bigcap_{i=1}^{m} G_{\{\pi_i\}} = 1.$$

**Distinguishing number $D(G)$:** The minimal size of a dist. partition.

**Remark.** $D(\Gamma) = D(\mathrm{Aut}(\Gamma))$, so $D(D_{10}) = 3$ and $D(D_{2n}) = 2$ for $n \geqslant 6$.

## Examples

- $D(S_n) = n$; $D(A_n) = n - 1$.
- $D(G) = 1 \iff G = 1$.
- $G \neq 1$ is regular $\implies D(G) = 2$.

# Primitive groups

**Note.** The following statements are equivalent.

- $D(G) = 2$;
- $G$ has a regular orbit on the power set $\mathcal{P}(\Omega)$ of $\Omega$;
- $\exists\ \Delta \subseteq \Omega$ such that $G_{\{\Delta\}} = 1$.

# Primitive groups

**Note.** The following statements are equivalent.

- $D(G) = 2$;
- $G$ has a regular orbit on the power set $\mathcal{P}(\Omega)$ of $\Omega$;
- $\exists\ \Delta \subseteq \Omega$ such that $G_{\{\Delta\}} = 1$.

$G$ is called **primitive** if $G_\alpha$ is maximal in $G$.

# Primitive groups

**Note.** The following statements are equivalent.

- $D(G) = 2$;
- $G$ has a regular orbit on the power set $\mathcal{P}(\Omega)$ of $\Omega$;
- $\exists\ \Delta \subseteq \Omega$ such that $G_{\{\Delta\}} = 1$.

$G$ is called **primitive** if $G_\alpha$ is maximal in $G$.

**Theorem (Cameron, Neumann & Saxl, 1984; Seress, 1997)**

$G \neq A_n, S_n$ primitive $\implies D(G) = 2$

# Primitive groups

**Note.** The following statements are equivalent.

- $D(G) = 2$;
- $G$ has a regular orbit on the power set $\mathcal{P}(\Omega)$ of $\Omega$;
- $\exists\ \Delta \subseteq \Omega$ such that $G_{\{\Delta\}} = 1$.

$G$ is called **primitive** if $G_\alpha$ is maximal in $G$.

**Theorem (Cameron, Neumann & Saxl, 1984; Seress, 1997)**

$G \neq A_n, S_n$ primitive $\implies D(G) = 2$, with 43 exceptions of degree $\leqslant 32$.

# Primitive groups

**Note.** The following statements are equivalent.

- $D(G) = 2$;
- $G$ has a regular orbit on the power set $\mathcal{P}(\Omega)$ of $\Omega$;
- $\exists\ \Delta \subseteq \Omega$ such that $G_{\{\Delta\}} = 1$.

$G$ is called **primitive** if $G_\alpha$ is maximal in $G$.

**Theorem (Cameron, Neumann & Saxl, 1984; Seress, 1997)**

$G \neq A_n, S_n$ primitive $\implies D(G) = 2$, with 43 exceptions of degree $\leqslant 32$.

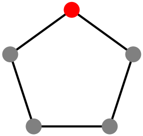
**Dolfi, 2000:** $D(G) \leqslant 4$ for each exception.

# Part II

Bases for permutation groups

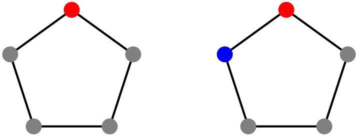
# Fixing sets

Which automorphisms of $\Gamma = \mathbf{C}_5$ survive if we "pin" each coloured vertex?



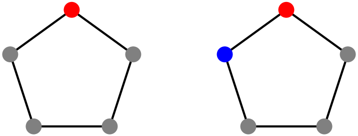
# Fixing sets

Which automorphisms of $\Gamma = \mathbf{C}_5$ survive if we "pin" each coloured vertex?



# Fixing sets

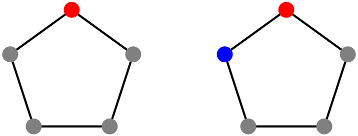
Which automorphisms of $\Gamma = \mathbf{C}_5$ survive if we "pin" each coloured vertex?



**Fixing (determining) set:** A subset $\Delta \subseteq V\Gamma$ such that $\bigcap_{\alpha \in \Delta} \mathrm{Aut}(\Gamma)_\alpha = 1$.

# Fixing sets

Which automorphisms of $\Gamma = \mathbf{C}_5$ survive if we "pin" each coloured vertex?

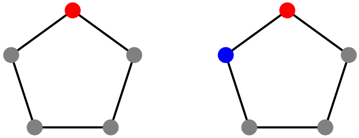


**Fixing (determining) set:** A subset $\Delta \subseteq V\Gamma$ such that $\bigcap_{\alpha \in \Delta} \mathrm{Aut}(\Gamma)_\alpha = 1$.

**Fixing (determining) number** $\mathrm{fix}(\Gamma)$**:** The minimal size of such a $\Delta$.

# Fixing sets

Which automorphisms of $\Gamma = \mathbf{C}_5$ survive if we "pin" each coloured vertex?



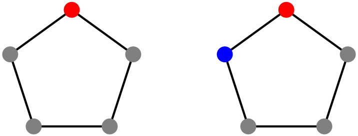
**Fixing (determining) set:** A subset $\Delta \subseteq V\Gamma$ such that $\bigcap_{\alpha\in\Delta} \mathrm{Aut}(\Gamma)_\alpha = 1$.

**Fixing (determining) number** $\mathrm{fix}(\Gamma)$**:** The minimal size of such a $\Delta$.

- $\Gamma = \mathbf{K}_n \implies \mathrm{fix}(\Gamma) = n - 1$.

# Fixing sets

Which automorphisms of $\Gamma = \mathbf{C}_5$ survive if we "pin" each coloured vertex?



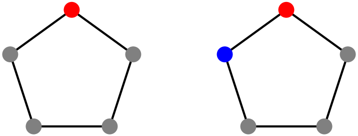
**Fixing (determining) set:** A subset $\Delta \subseteq V\Gamma$ such that $\bigcap_{\alpha \in \Delta} \mathrm{Aut}(\Gamma)_\alpha = 1$.

**Fixing (determining) number** $\mathrm{fix}(\Gamma)$**:** The minimal size of such a $\Delta$.

- $\Gamma = \mathbf{K}_n \implies \mathrm{fix}(\Gamma) = n - 1$.
- $\Gamma = \mathbf{C}_n \implies \mathrm{fix}(\Gamma) = 2$.

# Fixing sets

Which automorphisms of $\Gamma = \mathbf{C}_5$ survive if we "pin" each coloured vertex?



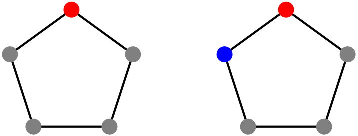
**Fixing (determining) set:** A subset $\Delta \subseteq V\Gamma$ such that $\bigcap_{\alpha \in \Delta} \mathrm{Aut}(\Gamma)_\alpha = 1$.

**Fixing (determining) number** $\mathrm{fix}(\Gamma)$**:** The minimal size of such a $\Delta$.

- $\Gamma = \mathbf{K}_n \implies \mathrm{fix}(\Gamma) = n - 1$.
- $\Gamma = \mathbf{C}_n \implies \mathrm{fix}(\Gamma) = 2$.
- $\mathrm{Aut}(\Gamma) = 1 \iff \mathrm{fix}(\Gamma) = 0$.

# Fixing sets

Which automorphisms of $\Gamma = \mathbf{C}_5$ survive if we "pin" each coloured vertex?



**Fixing (determining) set:** A subset $\Delta \subseteq V\Gamma$ such that $\bigcap_{\alpha \in \Delta} \mathrm{Aut}(\Gamma)_\alpha = 1$.

**Fixing (determining) number** $\mathrm{fix}(\Gamma)$**:** The minimal size of such a $\Delta$.

- $\Gamma = \mathbf{K}_n \implies \mathrm{fix}(\Gamma) = n - 1$.
- $\Gamma = \mathbf{C}_n \implies \mathrm{fix}(\Gamma) = 2$.
- $\mathrm{Aut}(\Gamma) = 1 \iff \mathrm{fix}(\Gamma) = 0$.
- $D(\Gamma) \leqslant \mathrm{fix}(\Gamma) + 1$.

# Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group.

**Base:** A subset $\Delta \subseteq \Omega$ such that $G_{(\Delta)} = \bigcap_{\alpha \in \Delta} G_\alpha = 1$.

# Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group.

**Base:** A subset $\Delta \subseteq \Omega$ such that $G_{(\Delta)} = \bigcap_{\alpha \in \Delta} G_\alpha = 1$.

**Base size** $b(G)$**:** The minimal size of a base for $G$.

# Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group.

**Base:** A subset $\Delta \subseteq \Omega$ such that $G_{(\Delta)} = \bigcap_{\alpha \in \Delta} G_\alpha = 1$.

**Base size $b(G)$:** The minimal size of a base for $G$.

**Remark.** $\mathrm{fix}(\Gamma) = b(\mathrm{Aut}(\Gamma))$.

## Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group.

**Base:** A subset $\Delta \subseteq \Omega$ such that $G_{(\Delta)} = \bigcap_{\alpha \in \Delta} G_\alpha = 1$.

**Base size $b(G)$:** The minimal size of a base for $G$.

**Remark.** $\mathrm{fix}(\Gamma) = b(\mathrm{Aut}(\Gamma))$.

- $G = S_n$, $\Omega = \{1, \ldots, n\} \implies b(G) = n - 1$.

# Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group.

**Base:** A subset $\Delta \subseteq \Omega$ such that $G_{(\Delta)} = \bigcap_{\alpha \in \Delta} G_\alpha = 1$.

**Base size $b(G)$:** The minimal size of a base for $G$.

**Remark.** $\mathrm{fix}(\Gamma) = b(\mathrm{Aut}(\Gamma))$.

- $G = S_n$, $\Omega = \{1, \ldots, n\} \implies b(G) = n - 1$.
- $G = D_{2n}$ $(n \geqslant 3)$, $\Omega = \{1, \ldots, n\} \implies b(G) = 2$.

# Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group.

**Base:** A subset $\Delta \subseteq \Omega$ such that $G_{(\Delta)} = \bigcap_{\alpha \in \Delta} G_\alpha = 1$.

**Base size $b(G)$:** The minimal size of a base for $G$.

**Remark.** $\mathrm{fix}(\Gamma) = b(\mathrm{Aut}(\Gamma))$.

- $G = S_n$, $\Omega = \{1, \ldots, n\} \implies b(G) = n - 1$.
- $G = D_{2n}$ $(n \geqslant 3)$, $\Omega = \{1, \ldots, n\} \implies b(G) = 2$.
- $b(G) = 0 \iff G = 1$.

# Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group.

**Base:** A subset $\Delta \subseteq \Omega$ such that $G_{(\Delta)} = \bigcap_{\alpha \in \Delta} G_\alpha = 1$.

**Base size $b(G)$:** The minimal size of a base for $G$.

**Remark.** $\mathrm{fix}(\Gamma) = b(\mathrm{Aut}(\Gamma))$.

- $G = S_n$, $\Omega = \{1, \ldots, n\} \implies b(G) = n - 1$.
- $G = D_{2n}$ $(n \geqslant 3)$, $\Omega = \{1, \ldots, n\} \implies b(G) = 2$.
- $b(G) = 0 \iff G = 1$.
- $D(G) \leqslant b(G) + 1$.

# Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group.

**Base:** A subset $\Delta \subseteq \Omega$ such that $G_{(\Delta)} = \bigcap_{\alpha \in \Delta} G_\alpha = 1$.

**Base size $b(G)$:** The minimal size of a base for $G$.

**Remark.** $\mathrm{fix}(\Gamma) = b(\mathrm{Aut}(\Gamma))$.

- $G = S_n$, $\Omega = \{1, \ldots, n\} \implies b(G) = n - 1$.
- $G = D_{2n}$ $(n \geqslant 3)$, $\Omega = \{1, \ldots, n\} \implies b(G) = 2$.
- $b(G) = 0 \iff G = 1$.
- $D(G) \leqslant b(G) + 1$.
- $G = \mathrm{GL}_d(q)$, $\Omega = \mathbb{F}_q^d \setminus \{0\} \implies b(G) = d$.

# Bases

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group.

**Base:** A subset $\Delta \subseteq \Omega$ such that $G_{(\Delta)} = \bigcap_{\alpha \in \Delta} G_\alpha = 1$.

**Base size $b(G)$:** The minimal size of a base for $G$.

**Remark.** $\mathrm{fix}(\Gamma) = b(\mathrm{Aut}(\Gamma))$.

- $G = S_n$, $\Omega = \{1, \ldots, n\} \implies b(G) = n - 1$.
- $G = D_{2n}$ $(n \geqslant 3)$, $\Omega = \{1, \ldots, n\} \implies b(G) = 2$.
- $b(G) = 0 \iff G = 1$.
- $D(G) \leqslant b(G) + 1$.
- $G = \mathrm{GL}_d(q)$, $\Omega = \mathbb{F}_q^d \setminus \{0\} \implies b(G) = d$.

  **Klavžar, Wong & Zhu, 2006:** $D(G) = 2$ if $\mathbb{F}_q^d \neq \mathbb{F}_2^2, \mathbb{F}_2^3, \mathbb{F}_4^2$ or $\mathbb{F}_3^2$.

# Base sizes

**Observation:** If $\Delta$ is a base and $x, y \in G$,

# Base sizes

**Observation:** If $\Delta$ is a base and $x, y \in G$, then

$$\alpha^x = \alpha^y \text{ for all } \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha = G_{(\Delta)}$$

# Base sizes

**Observation:** If $\Delta$ is a base and $x, y \in G$, then

$$\alpha^x = \alpha^y \text{ for all } \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha = G_{(\Delta)} \iff x = y.$$

# Base sizes

**Observation:** If $\Delta$ is a base and $x, y \in G$, then

$$\alpha^x = \alpha^y \text{ for all } \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha = G_{(\Delta)} \iff x = y.$$

That is, each group element is uniquely determined by its action on $\Delta$.

# Base sizes

**Observation:** If $\Delta$ is a base and $x, y \in G$, then

$$\alpha^x = \alpha^y \text{ for all } \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha = G_{(\Delta)} \iff x = y.$$

That is, each group element is uniquely determined by its action on $\Delta$.

- $|G| \leqslant n^{b(G)}$

# Base sizes

**Observation:** If $\Delta$ is a base and $x, y \in G$, then

$$\alpha^x = \alpha^y \text{ for all } \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha = G_{(\Delta)} \iff x = y.$$

That is, each group element is uniquely determined by its action on $\Delta$.

- $|G| \leqslant n^{b(G)}$
- A small base $\Delta$ provides an efficient way to store the elements of $G$, using $|\Delta|$-tuples rather than $|\Omega|$-tuples.

# Base sizes

**Observation:** If $\Delta$ is a base and $x, y \in G$, then

$$\alpha^x = \alpha^y \text{ for all } \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha = G_{(\Delta)} \iff x = y.$$

That is, each group element is uniquely determined by its action on $\Delta$.

- $|G| \leqslant n^{b(G)}$
- A small base $\Delta$ provides an efficient way to store the elements of $G$, using $|\Delta|$-tuples rather than $|\Omega|$-tuples.

**Question:** How small is $b(G)$?

# Base sizes

**Observation:** If $\Delta$ is a base and $x, y \in G$, then

$$\alpha^x = \alpha^y \text{ for all } \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha = G_{(\Delta)} \iff x = y.$$

That is, each group element is uniquely determined by its action on $\Delta$.

- $|G| \leqslant n^{b(G)}$
- A small base $\Delta$ provides an efficient way to store the elements of $G$, using $|\Delta|$-tuples rather than $|\Omega|$-tuples.

**Question:** How small is $b(G)$?

**Note.** $b(G) = 1 \iff G \neq 1$ is regular.

# Part III

## The base-two project

# Base-two groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be transitive of degree $n$ with point stabiliser $H = G_\alpha$.

# Base-two groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be transitive of degree $n$ with point stabiliser $H = G_\alpha$.

$$\begin{aligned} b(G) = 2 &\iff H \neq 1 \text{ has a regular orbit on } \Omega \\ &\iff H \neq 1 \;\&\; H \cap H^g = 1 \text{ for some } g \in G \end{aligned}$$

# Base-two groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be transitive of degree $n$ with point stabiliser $H = G_\alpha$.

$$\begin{aligned} b(G) = 2 &\iff H \neq 1 \text{ has a regular orbit on } \Omega \\ &\iff H \neq 1 \ \& \ H \cap H^g = 1 \text{ for some } g \in G \end{aligned}$$

**Recall.** $G$ is called **primitive** if $H$ is maximal in $G$.

# Base-two groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be transitive of degree $n$ with point stabiliser $H = G_\alpha$.

$$b(G) = 2 \iff H \neq 1 \text{ has a regular orbit on } \Omega$$
$$\iff H \neq 1 \ \& \ H \cap H^g = 1 \text{ for some } g \in G$$

**Recall.** $G$ is called **primitive** if $H$ is maximal in $G$.

**Problem.** Classify the finite primitive groups $G$ with $b(G) = 2$.

# Base-two groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be transitive of degree $n$ with point stabiliser $H = G_\alpha$.

$$\begin{aligned} b(G) = 2 &\iff H \neq 1 \text{ has a regular orbit on } \Omega \\ &\iff H \neq 1 \ \&\ H \cap H^g = 1 \text{ for some } g \in G \end{aligned}$$

**Recall.** $G$ is called **primitive** if $H$ is maximal in $G$.

**Problem.** Classify the finite primitive groups $G$ with $b(G) = 2$.

**Example**

$p$ prime, $G = D_{2p}$ and $\Omega = \{1, \ldots, p\} \implies G$ primitive and $b(G) = 2$.

# Probabilistic methods

Let

$$Q(G) = \frac{|\{(\alpha, \beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{|\Omega|^2}$$

be the probability that a random pair in $\Omega$ is **not** a base for $G$.

# Probabilistic methods

Let

$$Q(G) = \frac{|\{(\alpha, \beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{|\Omega|^2}$$

be the probability that a random pair in $\Omega$ is **not** a base for $G$.

**Note.** $\{\alpha, \beta\}$ is not a base iff there exists $x \in G_\alpha \cap G_\beta$ of prime order.

# Probabilistic methods

Let

$$Q(G) = \frac{|\{(\alpha,\beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{|\Omega|^2}$$

be the probability that a random pair in $\Omega$ is **not** a base for $G$.

**Note.** $\{\alpha,\beta\}$ is not a base iff there exists $x \in G_\alpha \cap G_\beta$ of prime order.

The probability that a random pair is fixed by $x \in G$ is $\mathrm{fpr}(x)^2$, where

$$\mathrm{fpr}(x) = \frac{|C_\Omega(x)|}{|\Omega|} = \frac{|x^G \cap H|}{|x^G|}$$

is the fixed point ratio of $x$.

# Probabilistic methods

Let

$$Q(G) = \frac{|\{(\alpha,\beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{|\Omega|^2}$$

be the probability that a random pair in $\Omega$ is **not** a base for $G$.

**Note.** $\{\alpha, \beta\}$ is not a base iff there exists $x \in G_\alpha \cap G_\beta$ of prime order.

The probability that a random pair is fixed by $x \in G$ is $\mathrm{fpr}(x)^2$, where

$$\mathrm{fpr}(x) = \frac{|C_\Omega(x)|}{|\Omega|} = \frac{|x^G \cap H|}{|x^G|}$$

is the fixed point ratio of $x$. Therefore,

$$Q(G) \leqslant \sum_{x \in \mathcal{P}} \mathrm{fpr}(x)^2 = \sum_i |x_i^G| \,\mathrm{fpr}(x)^2 =: \widehat{Q}(G),$$

where $\mathcal{P} = \bigcup_i x_i^G$ is the set of elements of prime order in $G$.

# Probabilistic methods

Let

$$Q(G) = \frac{|\{(\alpha,\beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{|\Omega|^2}$$

be the probability that a random pair in $\Omega$ is **not** a base for $G$.

**Note.** $\{\alpha,\beta\}$ is not a base iff there exists $x \in G_\alpha \cap G_\beta$ of prime order.

The probability that a random pair is fixed by $x \in G$ is $\mathrm{fpr}(x)^2$, where

$$\mathrm{fpr}(x) = \frac{|C_\Omega(x)|}{|\Omega|} = \frac{|x^G \cap H|}{|x^G|}$$

is the fixed point ratio of $x$. Therefore,

$$Q(G) \leqslant \sum_{x \in \mathcal{P}} \mathrm{fpr}(x)^2 = \sum_i |x_i^G|\,\mathrm{fpr}(x)^2 =: \widehat{Q}(G),$$

where $\mathcal{P} = \bigcup_i x_i^G$ is the set of elements of prime order in $G$.

**Probabilistic method:** $\widehat{Q}(G) < 1 \implies b(G) \leqslant 2.$

# An example

We have $b(G) \leqslant 2$ if

$$\widehat{Q}(G) = \sum_i |x_i^G| \operatorname{fpr}(x)^2 = \sum_i \frac{|x_i^G \cap H|^2}{|x_i^G|} < 1,$$

where $\mathcal{P} = \bigcup_i x_i^G$ is the set of elements of prime order in $G$.

# An example

We have $b(G) \leqslant 2$ if

$$\widehat{Q}(G) = \sum_i |x_i^G| \operatorname{fpr}(x)^2 = \sum_i \frac{|x_i^G \cap H|^2}{|x_i^G|} < 1,$$

where $\mathcal{P} = \bigcup_i x_i^G$ is the set of elements of prime order in $G$.

**Note.** If $\sum_i |x_i^G \cap H| \leqslant A$ and $|x_i^G| \geqslant B$ for all $i$, then $\widehat{Q}(G) \leqslant A^2/B$.

# An example

We have $b(G) \leqslant 2$ if

$$\widehat{Q}(G) = \sum_i |x_i^G| \,\mathrm{fpr}(x)^2 = \sum_i \frac{|x_i^G \cap H|^2}{|x_i^G|} < 1,$$

where $\mathcal{P} = \bigcup_i x_i^G$ is the set of elements of prime order in $G$.

**Note.** If $\sum_i |x_i^G \cap H| \leqslant A$ and $|x_i^G| \geqslant B$ for all $i$, then $\widehat{Q}(G) \leqslant A^2/B$.

## Example

Let $p \geqslant 11$ be a prime, $G = S_p$ and $H = \mathrm{AGL}_1(p)$.

# An example

We have $b(G) \leqslant 2$ if

$$\widehat{Q}(G) = \sum_i |x_i^G| \operatorname{fpr}(x)^2 = \sum_i \frac{|x_i^G \cap H|^2}{|x_i^G|} < 1,$$

where $\mathcal{P} = \bigcup_i x_i^G$ is the set of elements of prime order in $G$.

**Note.** If $\sum_i |x_i^G \cap H| \leqslant A$ and $|x_i^G| \geqslant B$ for all $i$, then $\widehat{Q}(G) \leqslant A^2/B$.

## Example

Let $p \geqslant 11$ be a prime, $G = S_p$ and $H = \mathrm{AGL}_1(p)$. Note that we have $\sum_i |x_i^G \cap H| \leqslant |H| = p(p-1) =: A$.

# An example

We have $b(G) \leqslant 2$ if

$$\widehat{Q}(G) = \sum_i |x_i^G| \operatorname{fpr}(x)^2 = \sum_i \frac{|x_i^G \cap H|^2}{|x_i^G|} < 1,$$

where $\mathcal{P} = \bigcup_i x_i^G$ is the set of elements of prime order in $G$.

**Note.** If $\sum_i |x_i^G \cap H| \leqslant A$ and $|x_i^G| \geqslant B$ for all $i$, then $\widehat{Q}(G) \leqslant A^2/B$.

## Example

Let $p \geqslant 11$ be a prime, $G = S_p$ and $H = \mathrm{AGL}_1(p)$. Note that we have $\sum_i |x_i^G \cap H| \leqslant |H| = p(p-1) =: A$.

If $|x_i| = p$ then $|x_i^G| = (p-1)!$,

# An example

We have $b(G) \leqslant 2$ if

$$\widehat{Q}(G) = \sum_i |x_i^G| \, \mathrm{fpr}(x)^2 = \sum_i \frac{|x_i^G \cap H|^2}{|x_i^G|} < 1,$$

where $\mathcal{P} = \bigcup_i x_i^G$ is the set of elements of prime order in $G$.

**Note.** If $\sum_i |x_i^G \cap H| \leqslant A$ and $|x_i^G| \geqslant B$ for all $i$, then $\widehat{Q}(G) \leqslant A^2/B$.

## Example

Let $p \geqslant 11$ be a prime, $G = S_p$ and $H = \mathrm{AGL}_1(p)$. Note that we have $\sum_i |x_i^G \cap H| \leqslant |H| = p(p-1) =: A$.

If $|x_i| = p$ then $|x_i^G| = (p-1)!$, and if $x_i$ has cycle shape $[1, r^{(p-1)/r}]$ then

$$|x_i^G| = \frac{p!}{r^{(p-1)/r}(\frac{p-1}{r})!} \geqslant \frac{p!}{2^{(p-1)/2}(\frac{p-1}{2})!} =: B.$$

# An example

We have $b(G) \leqslant 2$ if

$$\widehat{Q}(G) = \sum_i |x_i^G| \, \mathrm{fpr}(x)^2 = \sum_i \frac{|x_i^G \cap H|^2}{|x_i^G|} < 1,$$

where $\mathcal{P} = \bigcup_i x_i^G$ is the set of elements of prime order in $G$.

**Note.** If $\sum_i |x_i^G \cap H| \leqslant A$ and $|x_i^G| \geqslant B$ for all $i$, then $\widehat{Q}(G) \leqslant A^2/B$.

## Example

Let $p \geqslant 11$ be a prime, $G = S_p$ and $H = \mathrm{AGL}_1(p)$. Note that we have $\sum_i |x_i^G \cap H| \leqslant |H| = p(p-1) =: A$.

If $|x_i| = p$ then $|x_i^G| = (p-1)!$, and if $x_i$ has cycle shape $[1, r^{(p-1)/r}]$ then

$$|x_i^G| = \frac{p!}{r^{(p-1)/r}(\frac{p-1}{r})!} \geqslant \frac{p!}{2^{(p-1)/2}(\frac{p-1}{2})!} =: B.$$

It is easy to see that $A^2/B \to 0$, so $\widehat{Q}(G) \to 0$ and $b(G) \leqslant 2$.

# Some results

The **O'Nan-Scott Theorem** divides finite primitive groups into 5 types.

# Some results

The **O'Nan-Scott Theorem** divides finite primitive groups into 5 types.

**Affine:** Partial classification in the setting $H/Z(H)$ is simple.

## Some results

The **O'Nan-Scott Theorem** divides finite primitive groups into 5 types.

**Affine:** Partial classification in the setting $H/Z(H)$ is simple.

**Twisted wreath type:** Partial results **(Fawcett, 2022)**

# Some results

The **O'Nan-Scott Theorem** divides finite primitive groups into 5 types.

**Affine:** Partial classification in the setting $H/Z(H)$ is simple.

**Twisted wreath type:** Partial results **(Fawcett, 2022)**

**Almost simple:** $T \trianglelefteqslant G \leqslant \mathrm{Aut}(T)$ for non-abelian simple $T$.

# Some results

The **O'Nan-Scott Theorem** divides finite primitive groups into 5 types.

**Affine:** Partial classification in the setting $H/Z(H)$ is simple.

**Twisted wreath type:** Partial results **(Fawcett, 2022)**

**Almost simple:** $T \trianglelefteqslant G \leqslant \mathrm{Aut}(T)$ for non-abelian simple $T$.

- **Burness et al., 2010/11:** $T$ alternating or sporadic ✓

# Some results

The **O'Nan-Scott Theorem** divides finite primitive groups into 5 types.

**Affine:** Partial classification in the setting $H/Z(H)$ is simple.

**Twisted wreath type:** Partial results **(Fawcett, 2022)**

**Almost simple:** $T \trianglelefteq G \leqslant \mathrm{Aut}(T)$ for non-abelian simple $T$.

- **Burness et al., 2010/11:** $T$ alternating or sporadic ✓
- $G$ Lie type: Partial answers

# Some results

The **O'Nan-Scott Theorem** divides finite primitive groups into 5 types.

**Affine:** Partial classification in the setting $H/Z(H)$ is simple.

**Twisted wreath type:** Partial results **(Fawcett, 2022)**

**Almost simple:** $T \trianglelefteq G \leqslant \mathrm{Aut}(T)$ for non-abelian simple $T$.

- **Burness et al., 2010/11:** $T$ alternating or sporadic ✓
- $G$ Lie type: Partial answers

**Product type:** $G \leqslant L \wr P$ with its product action on $\Sigma^k$, where $L \leqslant \mathrm{Sym}(\Sigma)$.

# Some results

The **O'Nan-Scott Theorem** divides finite primitive groups into 5 types.

**Affine:** Partial classification in the setting $H/Z(H)$ is simple.

**Twisted wreath type:** Partial results **(Fawcett, 2022)**

**Almost simple:** $T \trianglelefteq G \leqslant \mathrm{Aut}(T)$ for non-abelian simple $T$.

- **Burness et al., 2010/11:** $T$ alternating or sporadic ✓
- $G$ Lie type: Partial answers

**Product type:** $G \leqslant L \wr P$ with its product action on $\Sigma^k$, where $L \leqslant \mathrm{Sym}(\Sigma)$.

**Bailey & Cameron, 2011:**

$$b(L \wr P) = 2 \iff L_\sigma \text{ has at least } D(P) \text{ regular orbits on } \Sigma.$$

# Some results

The **O'Nan-Scott Theorem** divides finite primitive groups into 5 types.

**Affine:** Partial classification in the setting $H/Z(H)$ is simple.

**Twisted wreath type:** Partial results **(Fawcett, 2022)**

**Almost simple:** $T \trianglelefteqslant G \leqslant \mathrm{Aut}(T)$ for non-abelian simple $T$.

- **Burness et al., 2010/11:** $T$ alternating or sporadic ✓
- $G$ Lie type: Partial answers

**Product type:** $G \leqslant L \wr P$ with its product action on $\Sigma^k$, where $L \leqslant \mathrm{Sym}(\Sigma)$.

**Bailey & Cameron, 2011:**

$$b(L \wr P) = 2 \iff L_\sigma \text{ has at least } D(P) \text{ regular orbits on } \Sigma.$$

**Burness & H, 2022+:** Progress where $G < L \wr P$

# Diagonal type

**Diagonal type:** $T^k \trianglelefteqslant G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P$ is primitive or $P = A_2$.

# Diagonal type

**Diagonal type:** $T^k \trianglelefteq G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P$ is primitive or $P = A_2$.

Write $\Omega = T^k/D$, where $D = \{(t, \ldots, t) : t \in T\}$ and set $\alpha = D$, so

$$G_\alpha = G \cap \{(a, \ldots, a)\pi : a \in \mathrm{Aut}(T), \pi \in P\}.$$

# Diagonal type

**Diagonal type:** $T^k \trianglelefteqslant G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P$ is primitive or $P = A_2$.

Write $\Omega = T^k/D$, where $D = \{(t, \dots, t) : t \in T\}$ and set $\alpha = D$, so

$$G_\alpha = G \cap \{(a, \dots, a)\pi : a \in \mathrm{Aut}(T), \pi \in P\}.$$

**Note.** If $k > 32$ and $P \neq A_k, S_k$ then $D(P) = 2$, so there exists a distinguishing partition $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$ of **distinct sizes**.

# Diagonal type

**Diagonal type:** $T^k \trianglelefteqslant G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P$ is primitive or $P = A_2$.

Write $\Omega = T^k/D$, where $D = \{(t, \ldots, t) : t \in T\}$ and set $\alpha = D$, so

$$G_\alpha = G \cap \{(a, \ldots, a)\pi : a \in \mathrm{Aut}(T), \pi \in P\}.$$

**Note.** If $k > 32$ and $P \neq A_k, S_k$ then $D(P) = 2$, so there exists a distinguishing partition $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$ of **distinct sizes**.

Write $T = \langle x, y \rangle$ and $\beta = D(t_1, \ldots, t_k) \in \Omega$, where

$$t_i = 1 \text{ if } i \in \Delta_1,\ \ t_i = x \text{ if } i \in \Delta_2,\ \ t_i = y \text{ if } i \in \Delta_3.$$

# Diagonal type

**Diagonal type:** $T^k \trianglelefteqslant G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P$ is primitive or $P = A_2$.

Write $\Omega = T^k/D$, where $D = \{(t,\dots,t) : t \in T\}$ and set $\alpha = D$, so

$$G_\alpha = G \cap \{(a,\dots,a)\pi : a \in \mathrm{Aut}(T), \pi \in P\}.$$

**Note.** If $k > 32$ and $P \neq A_k, S_k$ then $D(P) = 2$, so there exists a distinguishing partition $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$ of **distinct sizes**.

Write $T = \langle x, y\rangle$ and $\beta = D(t_1,\dots,t_k) \in \Omega$, where

$$t_i = 1 \text{ if } i \in \Delta_1,\ t_i = x \text{ if } i \in \Delta_2,\ t_i = y \text{ if } i \in \Delta_3.$$

Suppose $g = (a,\dots,a)\pi \in G_\alpha \cap G_\beta$.

# Diagonal type

**Diagonal type:** $T^k \trianglelefteq G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P$ is primitive or $P = A_2$.

Write $\Omega = T^k/D$, where $D = \{(t, \dots, t) : t \in T\}$ and set $\alpha = D$, so

$$G_\alpha = G \cap \{(a, \dots, a)\pi : a \in \mathrm{Aut}(T), \pi \in P\}.$$

**Note.** If $k > 32$ and $P \neq A_k, S_k$ then $D(P) = 2$, so there exists a distinguishing partition $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$ of **distinct sizes**.

Write $T = \langle x, y \rangle$ and $\beta = D(t_1, \dots, t_k) \in \Omega$, where

$$t_i = 1 \text{ if } i \in \Delta_1,\ \ t_i = x \text{ if } i \in \Delta_2,\ \ t_i = y \text{ if } i \in \Delta_3.$$

Suppose $g = (a, \dots, a)\pi \in G_\alpha \cap G_\beta$. Then $\pi = 1$ since $\pi$ preserves $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$.

# Diagonal type

**Diagonal type:** $T^k \trianglelefteqslant G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P$ is primitive or $P = A_2$.

Write $\Omega = T^k/D$, where $D = \{(t, \dots, t) : t \in T\}$ and set $\alpha = D$, so

$$G_\alpha = G \cap \{(a, \dots, a)\pi : a \in \mathrm{Aut}(T), \pi \in P\}.$$

**Note.** If $k > 32$ and $P \neq A_k, S_k$ then $D(P) = 2$, so there exists a distinguishing partition $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$ of **distinct sizes**.

Write $T = \langle x, y \rangle$ and $\beta = D(t_1, \dots, t_k) \in \Omega$, where

$$t_i = 1 \text{ if } i \in \Delta_1, \;\; t_i = x \text{ if } i \in \Delta_2, \;\; t_i = y \text{ if } i \in \Delta_3.$$

Suppose $g = (a, \dots, a)\pi \in G_\alpha \cap G_\beta$. Then $\pi = 1$ since $\pi$ preserves $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$. Thus, $a \in C_{\mathrm{Aut}(T)}(x) \cap C_{\mathrm{Aut}(T)}(y) = 1$ and so $g = 1$.

# Diagonal type

**Diagonal type:** $T^k \trianglelefteq G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P$ is primitive or $P = A_2$.

Write $\Omega = T^k/D$, where $D = \{(t,\dots,t) : t \in T\}$ and set $\alpha = D$, so

$$G_\alpha = G \cap \{(a,\dots,a)\pi : a \in \mathrm{Aut}(T), \pi \in P\}.$$

**Note.** If $k > 32$ and $P \neq A_k, S_k$ then $D(P) = 2$, so there exists a distinguishing partition $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$ of **distinct sizes**.

Write $T = \langle x, y \rangle$ and $\beta = D(t_1,\dots,t_k) \in \Omega$, where

$$t_i = 1 \text{ if } i \in \Delta_1,\ \ t_i = x \text{ if } i \in \Delta_2,\ \ t_i = y \text{ if } i \in \Delta_3.$$

Suppose $g = (a,\dots,a)\pi \in G_\alpha \cap G_\beta$. Then $\pi = 1$ since $\pi$ preserves $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$. Thus, $a \in C_{\mathrm{Aut}(T)}(x) \cap C_{\mathrm{Aut}(T)}(y) = 1$ and so $g = 1$.

**Fawcett, 2013:** $P \neq A_k, S_k \implies b(G) = 2$.

# Diagonal type

**Diagonal type:** $T^k \trianglelefteqslant G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P$ is primitive or $P = A_2$.

Write $\Omega = T^k/D$, where $D = \{(t, \dots, t) : t \in T\}$ and set $\alpha = D$, so

$$G_\alpha = G \cap \{(a, \dots, a)\pi : a \in \mathrm{Aut}(T), \pi \in P\}.$$

**Note.** If $k > 32$ and $P \neq A_k, S_k$ then $D(P) = 2$, so there exists a distinguishing partition $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$ of **distinct sizes**.

Write $T = \langle x, y \rangle$ and $\beta = D(t_1, \dots, t_k) \in \Omega$, where

$$t_i = 1 \text{ if } i \in \Delta_1,\;\; t_i = x \text{ if } i \in \Delta_2,\;\; t_i = y \text{ if } i \in \Delta_3.$$

Suppose $g = (a, \dots, a)\pi \in G_\alpha \cap G_\beta$. Then $\pi = 1$ since $\pi$ preserves $[k] = \Delta_1 \cup \Delta_2 \cup \Delta_3$. Thus, $a \in C_{\mathrm{Aut}(T)}(x) \cap C_{\mathrm{Aut}(T)}(y) = 1$ and so $g = 1$.

**Fawcett, 2013:** $P \neq A_k, S_k \implies b(G) = 2$.

**H, 2022+:** If $P = A_k$, then $b(G) = 2 \iff 2 < k < |T|$.

# Part IV

The Saxl graph of a base-two group

# Saxl graphs

Assume $G \leqslant \mathrm{Sym}(\Omega)$ is transitive of degree $n$ and $b(G) = 2$.

**Burness & Giudici, 2020:** **Saxl graph** $\Sigma(G)$:

$$\text{vertices } \Omega, \text{ with } \alpha \sim \beta \iff \{\alpha, \beta\} \text{ is a base.}$$

# Saxl graphs

Assume $G \leqslant \mathrm{Sym}(\Omega)$ is transitive of degree $n$ and $b(G) = 2$.

**Burness & Giudici, 2020:** **Saxl graph** $\Sigma(G)$:

$$\text{vertices } \Omega, \text{ with } \alpha \sim \beta \iff \{\alpha, \beta\} \text{ is a base.}$$

## Example

- $G$ is **Frobenius** $\iff \Sigma(G) \cong \mathbf{K}_n$.

# Saxl graphs

Assume $G \leqslant \mathrm{Sym}(\Omega)$ is transitive of degree $n$ and $b(G) = 2$.

**Burness & Giudici, 2020:** **Saxl graph** $\Sigma(G)$:

$$\text{vertices } \Omega, \text{ with } \alpha \sim \beta \iff \{\alpha, \beta\} \text{ is a base.}$$

## Example

- $G$ is **Frobenius** $\iff \Sigma(G) \cong \mathbf{K}_n$.

  e.g. $G = D_{2p}$ and $n = p$ prime.

# Saxl graphs

Assume $G \leqslant \mathrm{Sym}(\Omega)$ is transitive of degree $n$ and $b(G) = 2$.

**Burness & Giudici, 2020:** **Saxl graph** $\Sigma(G)$:

$$\text{vertices } \Omega, \text{ with } \alpha \sim \beta \iff \{\alpha, \beta\} \text{ is a base.}$$

## Example

- $G$ is **Frobenius** $\iff \Sigma(G) \cong \mathbf{K}_n$.

  e.g. $G = D_{2p}$ and $n = p$ prime.

- $G = \mathbb{Z}_m \wr \mathbb{Z}_2$ and $n = 2m \implies \Sigma(G) \cong \mathbf{K}_{m,m}$.

# Saxl graphs

Assume $G \leqslant \mathrm{Sym}(\Omega)$ is transitive of degree $n$ and $b(G) = 2$.

**Burness & Giudici, 2020:** **Saxl graph** $\Sigma(G)$:

$$\text{vertices } \Omega, \text{ with } \alpha \sim \beta \iff \{\alpha, \beta\} \text{ is a base.}$$

## Example

- $G$ is **Frobenius** $\iff \Sigma(G) \cong \mathbf{K}_n$.

  e.g. $G = D_{2p}$ and $n = p$ prime.

- $G = \mathbb{Z}_m \wr \mathbb{Z}_2$ and $n = 2m \implies \Sigma(G) \cong \mathbf{K}_{m,m}$.
- $G = \mathrm{GL}_2(q)$ and $\Omega = \mathbb{F}_q^2 \setminus \{0\}$.

# Saxl graphs

Assume $G \leqslant \mathrm{Sym}(\Omega)$ is transitive of degree $n$ and $b(G) = 2$.

**Burness & Giudici, 2020:** **Saxl graph** $\Sigma(G)$:

$$\text{vertices } \Omega, \text{ with } \alpha \sim \beta \iff \{\alpha, \beta\} \text{ is a base.}$$

## Example

- $G$ is **Frobenius** $\iff \Sigma(G) \cong \mathbf{K}_n$.

  e.g. $G = D_{2p}$ and $n = p$ prime.

- $G = \mathbb{Z}_m \wr \mathbb{Z}_2$ and $n = 2m \implies \Sigma(G) \cong \mathbf{K}_{m,m}$.

- $G = \mathrm{GL}_2(q)$ and $\Omega = \mathbb{F}_q^2 \setminus \{0\}$.

  **Note.** $\{\alpha, \beta\}$ is a base $\iff$ $\{\alpha, \beta\}$ is a basis for $\mathbb{F}_q^2$.

# Saxl graphs

Assume $G \leqslant \mathrm{Sym}(\Omega)$ is transitive of degree $n$ and $b(G) = 2$.

**Burness & Giudici, 2020:** **Saxl graph** $\Sigma(G)$:

$$\text{vertices } \Omega, \text{ with } \alpha \sim \beta \iff \{\alpha, \beta\} \text{ is a base.}$$

## Example

- $G$ is **Frobenius** $\iff \Sigma(G) \cong \mathbf{K}_n$.

  e.g. $G = D_{2p}$ and $n = p$ prime.

- $G = \mathbb{Z}_m \wr \mathbb{Z}_2$ and $n = 2m \implies \Sigma(G) \cong \mathbf{K}_{m,m}$.

- $G = \mathrm{GL}_2(q)$ and $\Omega = \mathbb{F}_q^2 \setminus \{0\}$.

  **Note.** $\{\alpha, \beta\}$ is a base $\iff \{\alpha, \beta\}$ is a basis for $\mathbb{F}_q^2$.

  Hence, $\Sigma(G) \cong \mathbf{K}_{q^2-1} - (q+1)\mathbf{K}_{q-1}$ is **complete multipartite**.

# Further example

- $G = \mathrm{PGL}_2(q)$ and $\Omega = \{$2-subsets of $\{$1-spaces in $\mathbb{F}_q^2\}\}$.

# Further example

- $G = \mathrm{PGL}_2(q)$ and $\Omega = \{$2-subsets of $\{$1-spaces in $\mathbb{F}_q^2\}\}$.

  **Note.** $G_\alpha \cong D_{2(q-1)}$ and $\{\alpha, \beta\}$ is a base $\iff |\alpha \cap \beta| = 1$.

# Further example

- $G = \mathrm{PGL}_2(q)$ and $\Omega = \{\text{2-subsets of } \{\text{1-spaces in } \mathbb{F}_q^2\}\}$.

  **Note.** $G_\alpha \cong D_{2(q-1)}$ and $\{\alpha, \beta\}$ is a base $\iff |\alpha \cap \beta| = 1$.

  Hence, $\Sigma(G) \cong J(q+1, 2)$ is a **Johnson graph**.

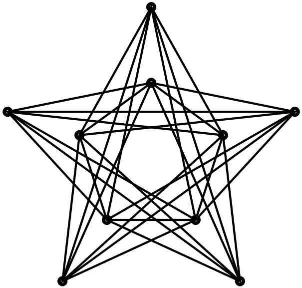
# Further example

- $G = \mathrm{PGL}_2(q)$ and $\Omega = \{$2-subsets of $\{$1-spaces in $\mathbb{F}_q^2\}\}$.

  **Note.** $G_\alpha \cong D_{2(q-1)}$ and $\{\alpha, \beta\}$ is a base $\iff |\alpha \cap \beta| = 1$.

  Hence, $\Sigma(G) \cong J(q+1, 2)$ is a **Johnson graph**.

  For example, when $q = 4$ we have the complement of the Petersen.



# Valency

Let $\Sigma(\alpha)$ be the set of neighbours of $\alpha$ in $\Sigma(G)$.

# Valency

Let $\Sigma(\alpha)$ be the set of neighbours of $\alpha$ in $\Sigma(G)$.

**Notes.**

- $\Sigma(\alpha)$ is the union of regular $G_\alpha$-orbits.

# Valency

Let $\Sigma(\alpha)$ be the set of neighbours of $\alpha$ in $\Sigma(G)$.

**Notes.**

- $\Sigma(\alpha)$ is the union of regular $G_\alpha$-orbits.
- $|\Sigma(\alpha)| = r(G)|G_\alpha|$, where $r(G)$ is the number of regular $G_\alpha$-orbits.

# Valency

Let $\Sigma(\alpha)$ be the set of neighbours of $\alpha$ in $\Sigma(G)$.

**Notes.**

- $\Sigma(\alpha)$ is the union of regular $G_\alpha$-orbits.
- $|\Sigma(\alpha)| = r(G)|G_\alpha|$, where $r(G)$ is the number of regular $G_\alpha$-orbits.

**Burness & Giudici, 2020:** $|\Sigma(\alpha)| = p$ is a prime iff the following holds:

- $G = \mathbb{Z}_p \wr \mathbb{Z}_2$, $n = 2p$ and $\Sigma(G) \cong \mathbf{K}_{p,p}$;
- $G = S_3$, $n = p + 1 = 3$ and $\Sigma(G) \cong \mathbf{K}_3$;
- $G = \mathrm{AGL}_1(2^f)$, $n = p + 1 = 2^f$ and $\Sigma(G) \cong \mathbf{K}_{p+1}$.

# Valency

Let $\Sigma(\alpha)$ be the set of neighbours of $\alpha$ in $\Sigma(G)$.

**Notes.**

- $\Sigma(\alpha)$ is the union of regular $G_\alpha$-orbits.
- $|\Sigma(\alpha)| = r(G)|G_\alpha|$, where $r(G)$ is the number of regular $G_\alpha$-orbits.

**Burness & Giudici, 2020:** $|\Sigma(\alpha)| = p$ is a prime iff the following holds:

- $G = \mathbb{Z}_p \wr \mathbb{Z}_2$, $n = 2p$ and $\Sigma(G) \cong \mathbf{K}_{p,p}$;
- $G = S_3$, $n = p + 1 = 3$ and $\Sigma(G) \cong \mathbf{K}_3$;
- $G = \mathrm{AGL}_1(2^f)$, $n = p + 1 = 2^f$ and $\Sigma(G) \cong \mathbf{K}_{p+1}$.

In particular, the Petersen graph is not a Saxl graph.

# Valency

Let $\Sigma(\alpha)$ be the set of neighbours of $\alpha$ in $\Sigma(G)$.

**Notes.**

- $\Sigma(\alpha)$ is the union of regular $G_\alpha$-orbits.
- $|\Sigma(\alpha)| = r(G)|G_\alpha|$, where $r(G)$ is the number of regular $G_\alpha$-orbits.

**Burness & Giudici, 2020:** $|\Sigma(\alpha)| = p$ is a prime iff the following holds:

- $G = \mathbb{Z}_p \wr \mathbb{Z}_2$, $n = 2p$ and $\Sigma(G) \cong \mathbf{K}_{p,p}$;
- $G = S_3$, $n = p + 1 = 3$ and $\Sigma(G) \cong \mathbf{K}_3$;
- $G = \mathrm{AGL}_1(2^f)$, $n = p + 1 = 2^f$ and $\Sigma(G) \cong \mathbf{K}_{p+1}$.

In particular, the Petersen graph is not a Saxl graph.

**Chen & H, 2022:** $G$ almost simple primitive and $|\Sigma(\alpha)|$ prime-power ✓

# Valency

Let $\Sigma(\alpha)$ be the set of neighbours of $\alpha$ in $\Sigma(G)$.

**Notes.**

- $\Sigma(\alpha)$ is the union of regular $G_\alpha$-orbits.
- $|\Sigma(\alpha)| = r(G)|G_\alpha|$, where $r(G)$ is the number of regular $G_\alpha$-orbits.

**Burness & Giudici, 2020:** $|\Sigma(\alpha)| = p$ is a prime iff the following holds:

- $G = \mathbb{Z}_p \wr \mathbb{Z}_2$, $n = 2p$ and $\Sigma(G) \cong \mathbf{K}_{p,p}$;
- $G = S_3$, $n = p + 1 = 3$ and $\Sigma(G) \cong \mathbf{K}_3$;
- $G = \mathrm{AGL}_1(2^f)$, $n = p + 1 = 2^f$ and $\Sigma(G) \cong \mathbf{K}_{p+1}$.

In particular, the Petersen graph is not a Saxl graph.

**Chen & H, 2022:** $G$ almost simple primitive and $|\Sigma(\alpha)|$ prime-power ✓

e.g. $q = 2^f + 1$, $G = \mathrm{PGL}_2(q)$, $G_\alpha = D_{2(q-1)}$ and $|\Sigma(\alpha)| = 2^{f+1}$.

# Orbital graphs

**Note.** $\Sigma(G)$ is the union of the $r(G)$ regular **orbital graphs** of $G$.

# Orbital graphs

**Note.** $\Sigma(G)$ is the union of the $r(G)$ regular **orbital graphs** of $G$.

Thus, $\Sigma(G)$ is a $G$-**orbital graph** $\iff$ $r(G) = 1$.

# Orbital graphs

**Note.** $\Sigma(G)$ is the union of the $r(G)$ regular **orbital graphs** of $G$.

Thus, $\Sigma(G)$ is a **$G$-orbital graph** $\iff r(G) = 1$.

**Problem.** Classify the finite primitive groups $G$ with $r(G) = 1$.

# Orbital graphs

**Note.** $\Sigma(G)$ is the union of the $r(G)$ regular **orbital graphs** of $G$.

Thus, $\Sigma(G)$ is a $G$-**orbital graph** $\iff r(G) = 1$.

**Problem.** Classify the finite primitive groups $G$ with $r(G) = 1$.

**Burness & H, 2021+:** $G$ almost simple and $G_\alpha$ soluble ✓

# Orbital graphs

**Note.** $\Sigma(G)$ is the union of the $r(G)$ regular **orbital graphs** of $G$.

Thus, $\Sigma(G)$ is a **$G$-orbital graph** $\iff r(G) = 1$.

> **Problem.** Classify the finite primitive groups $G$ with $r(G) = 1$.

**Burness & H, 2021+:** $G$ almost simple and $G_\alpha$ soluble ✓

e.g. $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$.

# Connectedness

**Note.** $G$ is primitive $\implies \Sigma(G)$ is connected.

# Connectedness

**Note.** $G$ is primitive $\implies \Sigma(G)$ is connected.

**Question:** What is the diameter of $\Sigma(G)$ if $G$ is primitive?

# Connectedness

**Note.** $G$ is primitive $\implies \Sigma(G)$ is connected.

**Question:** What is the diameter of $\Sigma(G)$ if $G$ is primitive?

**Conjecture (Burness & Giudici, 2020)**

$G$ primitive and $\alpha, \beta \in \Omega \implies \Sigma(\alpha) \cap \Sigma(\beta) \neq \emptyset$.

# Connectedness

**Note.** $G$ is primitive $\implies \Sigma(G)$ is connected.

**Question:** What is the diameter of $\Sigma(G)$ if $G$ is primitive?

**Conjecture (Burness & Giudici, 2020)**

$G$ primitive and $\alpha, \beta \in \Omega \implies \Sigma(\alpha) \cap \Sigma(\beta) \neq \emptyset$.

Recall that

$$Q(G) = 1 - \frac{r(G)|G_\alpha|}{|\Omega|} = 1 - \frac{|\Sigma(\alpha)|}{|\Omega|}$$

is the probability that a random pair in $\Omega$ is not a base for $G$.

# Connectedness

**Note.** $G$ is primitive $\implies \Sigma(G)$ is connected.

**Question:** What is the diameter of $\Sigma(G)$ if $G$ is primitive?

**Conjecture (Burness & Giudici, 2020)**

$G$ primitive and $\alpha, \beta \in \Omega \implies \Sigma(\alpha) \cap \Sigma(\beta) \neq \emptyset$.

Recall that

$$Q(G) = 1 - \frac{r(G)|G_\alpha|}{|\Omega|} = 1 - \frac{|\Sigma(\alpha)|}{|\Omega|}$$

is the probability that a random pair in $\Omega$ is not a base for $G$.

**Recall.** $Q(G) < 1 \iff b(G) \leqslant 2$.

# Connectedness

**Note.** $G$ is primitive $\implies$ $\Sigma(G)$ is connected.

**Question:** What is the diameter of $\Sigma(G)$ if $G$ is primitive?

**Conjecture (Burness & Giudici, 2020)**

$G$ primitive and $\alpha, \beta \in \Omega \implies \Sigma(\alpha) \cap \Sigma(\beta) \neq \emptyset$.

Recall that

$$Q(G) = 1 - \frac{r(G)|G_\alpha|}{|\Omega|} = 1 - \frac{|\Sigma(\alpha)|}{|\Omega|}$$

is the probability that a random pair in $\Omega$ is not a base for $G$.

**Recall.** $Q(G) < 1 \iff b(G) \leqslant 2$.

**Note.** $Q(G) < 1/2 \iff |\Sigma(\alpha)| > \frac{1}{2}|\Omega|$

# Connectedness

**Note.** $G$ is primitive $\implies \Sigma(G)$ is connected.

**Question:** What is the diameter of $\Sigma(G)$ if $G$ is primitive?

**Conjecture (Burness & Giudici, 2020)**

$G$ primitive and $\alpha, \beta \in \Omega \implies \Sigma(\alpha) \cap \Sigma(\beta) \neq \emptyset$.

Recall that

$$Q(G) = 1 - \frac{r(G)|G_\alpha|}{|\Omega|} = 1 - \frac{|\Sigma(\alpha)|}{|\Omega|}$$

is the probability that a random pair in $\Omega$ is not a base for $G$.

**Recall.** $Q(G) < 1 \iff b(G) \leqslant 2$.

**Note.** $Q(G) < 1/2 \iff |\Sigma(\alpha)| > \frac{1}{2}|\Omega| \implies \Sigma(\alpha) \cap \Sigma(\beta) \neq \emptyset$.

# Connectedness

**Note.** $G$ is primitive $\implies$ $\Sigma(G)$ is connected.

**Question:** What is the diameter of $\Sigma(G)$ if $G$ is primitive?

**Conjecture (Burness & Giudici, 2020)**

$G$ primitive and $\alpha, \beta \in \Omega \implies \Sigma(\alpha) \cap \Sigma(\beta) \neq \emptyset$.

Recall that

$$Q(G) = 1 - \frac{r(G)|G_\alpha|}{|\Omega|} = 1 - \frac{|\Sigma(\alpha)|}{|\Omega|}$$

is the probability that a random pair in $\Omega$ is not a base for $G$.

**Recall.** $Q(G) < 1 \iff b(G) \leqslant 2$.

**Note.** $Q(G) < 1/2 \iff |\Sigma(\alpha)| > \frac{1}{2}|\Omega| \implies \Sigma(\alpha) \cap \Sigma(\beta) \neq \emptyset$.

e.g. $p \geqslant 11$ prime, $(G, H) = (S_p, \mathrm{AGL}_1(p)) \implies Q(G) < 1/2$.

# Burness-Giudici conjecture

## Example

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $\Sigma(G) = J(q+1, 2)$ has the common neighbour property, though $Q(G) \to 1$ as $q \to \infty$.

# Burness-Giudici conjecture

**Example**

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $\Sigma(G) = J(q+1, 2)$ has the common neighbour property, though $Q(G) \to 1$ as $q \to \infty$.

Some other evidence:

- **Chen & Du 2020+; Burness & H, 2021+:** $\mathrm{soc}(G) = \mathrm{L}_2(q)$ ✓

# Burness-Giudici conjecture

**Example**

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $\Sigma(G) = J(q+1, 2)$ has the common neighbour property, though $Q(G) \to 1$ as $q \to \infty$.

Some other evidence:

- **Chen & Du 2020+; Burness & H, 2021+:** $\mathrm{soc}(G) = \mathrm{L}_2(q)$ ✓
- **Burness & H, 2021+:** $G$ almost simple with $G_\alpha$ soluble ✓

# Burness-Giudici conjecture

**Example**

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $\Sigma(G) = J(q+1, 2)$ has the common neighbour property, though $Q(G) \to 1$ as $q \to \infty$.

Some other evidence:

- **Chen & Du 2020+; Burness & H, 2021+:** $\mathrm{soc}(G) = \mathrm{L}_2(q)$ ✓
- **Burness & H, 2021+:** $G$ almost simple with $G_\alpha$ soluble ✓

**Recall.** $\Sigma(\alpha)$ is the union of regular $G_\alpha$-orbits.

# Burness-Giudici conjecture

**Example**

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $\Sigma(G) = J(q+1,2)$ has the common neighbour property, though $Q(G) \to 1$ as $q \to \infty$.

Some other evidence:

- **Chen & Du 2020+; Burness & H, 2021+:** $\mathrm{soc}(G) = \mathrm{L}_2(q)$ ✓
- **Burness & H, 2021+:** $G$ almost simple with $G_\alpha$ soluble ✓

**Recall.** $\Sigma(\alpha)$ is the union of regular $G_\alpha$-orbits.

**BG conjecture:** $\Sigma(\alpha)$ meets the union of regular $G_\beta$-orbits.

# Burness-Giudici conjecture

**Example**

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $\Sigma(G) = J(q+1, 2)$ has the common neighbour property, though $Q(G) \to 1$ as $q \to \infty$.

Some other evidence:

- **Chen & Du 2020+; Burness & H, 2021+:** $\mathrm{soc}(G) = \mathrm{L}_2(q)$ ✓
- **Burness & H, 2021+:** $G$ almost simple with $G_\alpha$ soluble ✓

**Recall.** $\Sigma(\alpha)$ is the union of regular $G_\alpha$-orbits.

**BG conjecture:** $\Sigma(\alpha)$ meets the union of regular $G_\beta$-orbits.

**Conjecture (Burness & H, 2022+)**

$G$ primitive and $\alpha, \beta \in \Omega \implies \Sigma(\alpha)$ meets **every** regular $G_\beta$-orbit.

# Future work

- Other invariants of the Saxl graph

# Future work

- Other invariants of the Saxl graph

  **Burness & H, 2021+:** Results on clique and independence numbers

# Future work

- Other invariants of the Saxl graph

  **Burness & H, 2021+:** Results on clique and independence numbers

- Generalisations of Saxl graphs

# Future work

- Other invariants of the Saxl graph

  **Burness & H, 2021+:** Results on clique and independence numbers

- Generalisations of Saxl graphs

- Bases for affine groups

# Future work

- Other invariants of the Saxl graph

  **Burness & H, 2021+:** Results on clique and independence numbers

- Generalisations of Saxl graphs

- Bases for affine groups

  **Seress 1996:** $b(G) \leqslant 4$ if $G$ is primitive and soluble

# Future work

- Other invariants of the Saxl graph

  **Burness & H, 2021+:** Results on clique and independence numbers

- Generalisations of Saxl graphs

- Bases for affine groups

  **Seress 1996:** $b(G) \leqslant 4$ if $G$ is primitive and soluble

- Distinguishing numbers for transitive groups

# Future work

- Other invariants of the Saxl graph

  **Burness & H, 2021+:** Results on clique and independence numbers

- Generalisations of Saxl graphs

- Bases for affine groups

  **Seress 1996:** $b(G) \leqslant 4$ if $G$ is primitive and soluble

- Distinguishing numbers for transitive groups

  **Seress 1996:** $D(G) \leqslant 5$ if $G$ is soluble

**Thank you!**