Regular subgroups of primitive groups

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Groups, Graphs and Combinatorics

(on the occasion of Professor Cai Heng Li's 65th birthday)

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Primitive groups

Problem. Classify the transitive subgroups of primitive groups.

The **O'Nan-Scott theorem** divides finite primitive groups into 5 types.

- Affine (HA)
- Almost simple (AS)
- Diagonal type (HS & SD)
- Product type (HC, CD & PA)
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Problem. **Regular** or **soluble** transitive subgroups of **diagonal type** groups?

Holomorph simple groups

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$$T \trianglelefteqslant AB = AT = BT \leqslant \mathrm{Aut}(T).$$

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Example

$T = A_{q+1}$, $K \cong (A_{q-2} \times \mathrm{PSL}_2(q)).2$ associated to $S_{q+1} = S_{q-2}\,\mathrm{PGL}_2(q)$.

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Conversely, if $T = AB$, then there exists a transitive subgroup of G isomorphic to $A \times B$.

Regular subgroups of holomorph simple groups

Now assume K is a **regular** subgroup of $\mathrm{Hol}(T)$ w.r.t. the factorisation

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Note. A composition factor of A (or B) is a composition factor of K.

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Example

$T = G_2(q) = AB$ with $q = 3^f$, $A = \mathrm{SL}_3(q)$ and $B = \mathrm{SU}_3(q).2$. But

$$q^{16} \sim |\,\mathrm{SL}_3(q)| \cdot |\mathrm{SU}_3(q)| > |K| = |\,T| \sim q^{14},$$

so there is no regular subgroup K w.r.t. the factorisation $T = AB$.

Main results on holomorph simple groups

Theorem (H & Wang, 2025+)

For every finite simple group T, the **regular** subgroups of $\mathrm{Hol}(T)$ are determined, up to conjugacy.

Theorem (H & Wang, 2025+)

For every finite simple group T, the **soluble transitive** subgroups of $\mathrm{Hol}(T)$ are determined, up to conjugacy.

Application I: Hopf-Galois structure

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Theorem (H & Wang, 2025+)

The types of Hopf-Galois structures are determined on any Galois extension whose Galois group is finite simple.

Application II: Skew braces

Skew brace: A triple $(X, +, \circ)$, where $(X, +)$ and $(X, \circ)$ are groups, and

$$x \circ (y + z) = x \circ y - x + x \circ z \text{ for all } x, y, z \in X.$$

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Theorem (H & Wang, 2025+)

The skew braces with finite simple additive groups are classified.

Application III: Diagonal type groups

Let $k \geqslant 2$, T be a non-abelian simple group, and let

$$X := \{(t, \ldots, t) : t \in T\} \leqslant T^k.$$

Then $T^k \leqslant \mathrm{Sym}(\Omega)$ with $\Omega = [T^k : X]$ of size $|T|^{k-1}$.

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Theorem (H & Wang, 2025+)

If K is regular or soluble, then $k = 2$ and $K \leqslant \mathrm{Hol}(T)$ is determined.

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Assume $K \leqslant G$ is **regular** and assume K **contains** exactly $\ell \geqslant 1$ simple direct factors of T^k.

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The regular or soluble transitive subgroups of diagonal type groups are determined, up to conjugacy.

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Let $G = T^k{:}P$ be a primitive **twisted wreath product**, where $P \leqslant S_k$.

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Let $G = A_5^6{:}A_6$, $H = A_5^6{:}A_5 = (A_5^5{:}A_5) \times A_5$ and $L \cong A_5$ be the diagonal of the latter two A_5's. Then $K = A_5^5{:}L$ is a regular subgroup with $\ell = 5$.

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Spiga (personal communication): $\ell \geqslant k-2$ is best possible.

Future directions

- Transitive subgroups of diagonal type groups
- Regular subgroups of other primitive groups

  Remark. The only remaining families are the affine groups (hard!), the twisted wreath products and the product type groups.
- Soluble transitive groups of non-affine primitive groups

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- ....