Bases for permutation groups.

H.Y. Huang (UoB)

March 23^{rd}, 2023.

@ Imperial College London for JLAC.

1. Bases

Let $G \leq \mathrm{Sym}(\Omega)$, where $|\Omega| < \infty$ and G is transitive.

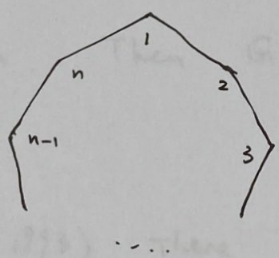
- Point stabiliser: $G_\alpha = \{ g \in G : \alpha^g = \alpha \}$.

Note. $\bigcap_{\alpha \in \Omega} G_\alpha = 1$.

Question: Any subset $\Delta \subseteq \Omega$ with $\bigcap_{\alpha \in \Delta} G_\alpha = 1$?

Example.

- $G = S_n$, $|\Omega| = n$, $\Delta = \{1, \cdots, n-1\}$. $\boxed{b(G) = n-1}$
- $G = D_{2n}$, $|\Omega| = n$, $\Delta = \{1, 2\}$, $\boxed{b(G) = 2}$



- $G = GL(V)$, $\Omega = V \setminus \{0\}$

  Δ contains a basis of V. $\boxed{b(G) = \dim V}$

Def - $\Delta \subseteq \Omega$ is called a base for G if $\bigcap_{\alpha \in \Delta} G_\alpha = 1$

- The base size of G, denoted $b(G)$, is the minimal size of a base for G.

Q1 Determine $b(G)$?

Q2. Bounds on $b(G)$?

Q3. Classify G with $b(G) = 2$?

Scanned with CamScanner

Lower bound

Let Δ be a base of size $b(G)$ and $x, y \in G$. Then

$$\alpha^x = \alpha^y \ \forall \alpha \in \Delta \iff x^{-1}y \in \bigcap_{\alpha \in \Delta} G_\alpha \iff x = y.$$

That is,

$$\text{elements of } G \xleftrightarrow{\text{one-to-one}} \text{images of } \Delta.$$

We have $|G| \leq |\Omega|^{b(G)} \Rightarrow b(G) \geq \log_{|\Omega|}|G|$.

Upper bound.

Write $\Delta = \{\alpha_1, \dots, \alpha_{b(G)}\}$ and $G^{(k)} = \bigcap_{i=1}^{k} G_{\alpha_i}$. Then

$$G \gneq G^{(1)} \gneq G^{(2)} \gneq \cdots \gneq G^{(b(G))} = 1.$$

Hence, $|G| \geq 2^{b(G)} \Rightarrow b(G) \leq \log_2|G|$.

2. Primitive groups

- "Primitive" = "transitive" + "G_α is maximal in G".

Example. $G = D_{2n}$. Then G is primitive $\iff$ n is a prime.

On Q2

Conjecture (Pyber, 1993) There exists an absolute constant c s.t.

$$b(G) \leq c \log_{|\Omega|}|G|.$$

for any primitive group $G \leq \mathrm{Sym}(\Omega)$.

- Duyan, Halasi and Maróti, 2018: Pyber's conj is true.
- Halasi, Liebeck and Maróti, 2019: $b(G) \leq 2\log_{|\Omega|}|G| + 24$.

Special cases:

- Seress, 1996: $b(G) \leq 4$ if G soluble
- Burness, 2021: $b(G) \leq 5$ if G_α soluble.

Scanned with CamScanner

On (Q1 and) Q3

Probabilistic method (Liebeck and Shalev, 1999):

$$Q(G) = \frac{|\{(\alpha, \beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{|\Omega|^2}$$

is the probability that a random pair of Ω is NOT a base.

Note $b(G) \leq 2 \iff Q(G) < 1$.

We have

$$Q(G) \leq \sum_{\substack{x \in G \\ |x| \text{ prime}}} \left(\frac{|x^G \cap G_\alpha|}{|G_\alpha|} \right)^2 =: \hat{Q}(G).$$

Note $\hat{Q}(G) < 1 \Rightarrow b(G) \leq 2$.

O'Nan-Scott

Finite primitive groups are divided into 5 types:

- Affine
- Almost simple
- Diagonal type
- Product type
- Twisted wreath

Scanned with CamScanner

3. Diagonal type

Let T be a non-abelian finite simple group and let

$$X = \{(x, \dots, x) : x \in T\} \leq T^k.$$

Then $T^k \leq \mathrm{Sym}(\Omega)$, where $\Omega = [T^k : X]$.

A group G is said to be <u>diagonal type</u> if

$$T^k \trianglelefteq G \leq W := N_{\mathrm{Sym}(\Omega)}(T^k) \cong T^k . (\mathrm{Out}(T) \times S_k)$$

Let D be a point stabiliser of W. Then

$$D = \{(\alpha, \dots, \alpha)\pi : \alpha \in \mathrm{Aut}(T),\ \pi \in S_k\}.$$

So $\Omega = [W : D] = \{D(t_1, \dots, t_k) : t_i \in T\}$, and

$$D(t_1, \dots, t_k)^{(\alpha, \dots, \alpha)\pi} = D(t^{\alpha}_{1\pi^{-1}}, \dots, t^{\alpha}_{k\pi^{-1}})$$

<u>Note</u> G induces $P_G \leq S_k$.

<u>Lemma</u> G is primitive $\iff$ P_G is primitive, or $k = 2$ and $P_G = 1$.

<u>Theorem</u> (Fawcett, 2013) $P_G \notin \{A_k, S_k\} \Rightarrow b(G) = 2$.

Assume $\mathrm{Hol}(T) = T : \mathrm{Aut}(T)$ acts on T by

$$t^{g\alpha} = (g^{-1}t)^{\alpha}$$

for any $t \in T$, $g \in T$, $\alpha \in \mathrm{Aut}(T)$.

Write $\mathrm{Hol}(T, S)$ for the setwise stabiliser of $S \subseteq T$.

<u>Lemma</u>. $b(G) = 2$ if $\exists S \subseteq T$ s.t. $|S| = k$ and $\mathrm{Hol}(T, S) = 1$.

<u>proof</u>. Let $S = \{t_1, \dots, t_k\}$, and $\Delta := \{D, D(t_1, \dots, t_k)\}$

Suppose $x \in G_{(\Delta)}$. Then $x = (\alpha, \dots, \alpha)\pi$ for some $\pi \in S_k$, $\alpha \in \mathrm{Aut}(T)$.

Then $D(t_1, \dots, t_k)^{(\alpha, \dots, \alpha)\pi} = D(t^{\alpha}_{1\pi^{-1}}, \dots, t^{\alpha}_{k\pi^{-1}}) = D(t_1, \dots, t_k)$

$\Rightarrow \exists g \in T$ s.t. $g^{-1} t^{\alpha}_{i\pi^{-1}} = t_{i\pi}$ $\forall i$

$\Rightarrow \{t_1, \dots, t_k\}^{g^{\alpha^{-1}}\alpha} = \{t_1, \dots, t_k\} \Rightarrow g = \alpha = 1 \Rightarrow \pi = 1 \Rightarrow x = 1$. $\square$

Scanned with CamScanner

Theorem (H, 2023+) If $3 \le k \le |T|-3$, then $\exists\, S \subseteq T$ s.t.

$|S| = k$ and $\mathrm{Hol}(T, S) = 1$.

Example. If $S = \{1, x, y\} \subseteq T$, then $\mathrm{Hol}(T, S) = 1$ if

$\langle x, y \rangle = T$ and $|x|, |y|, |x^{-1}y|$ are distinct.

Theorem (H, 2023+) $b(G) = 2 \iff$ one of the following holds.

(i) $P_G \notin \{A_k, S_k\}$

(ii) $3 \le k \le |T|-3$

(iii) $k \in \{|T|-2, |T|-1\}$ and $S_k \nleq G$.

Theorem (H, 2023+) Q1 for diagonal type primitive group is

completely answered.

Scanned with CamScanner