Permutations, bases and low rank groups.

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Consider $G = GL(V) \curvearrowright V \setminus \{0\}$.

(a) $\forall$ basis $\{v_1, \dots, v_n\}$ of V, $G_{v_1} \cap \cdots \cap G_{v_n} = 1$.

(b) The G_v-orbits are $\{av\}_{a \in \mathbb{F}^*}$, $V \setminus \langle v \rangle$.

Throughout, let $G \le \mathrm{Sym}(\Omega)$ be a transitive group and assume $|\Omega| < \infty$.

§1 Bases.

Base $\Delta \subseteq \Omega$ s.t. $\bigcap_{\alpha \in \Delta} G_\alpha = 1$.

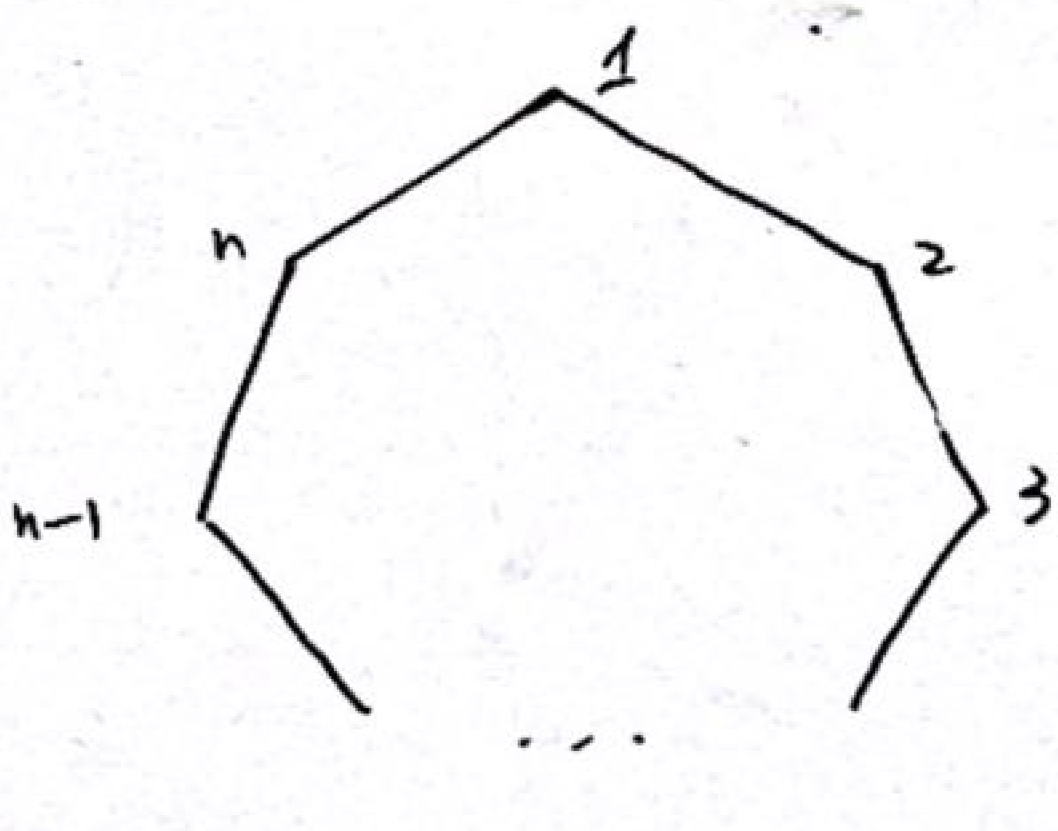
Base size $b(G)$: minimal size of a base for G.

Note $b(G) = \min\{k : G \text{ has a regular orbit on } \Omega^k\}$.

Let $r(G) := \#$ G-orbits on $\Omega^{b(G)}$.

Examples

- $G = GL(V)$, $\Omega = V \setminus \{0\}$
  Δ contains a basis of V. $b(G) = \dim V$ $r(G) = 1$.
- $G = S_n$, $|\Omega| = n$, $\Delta = \{1, \dots, n-1\}$. $b(G) = n-1$, $r(G) = 1$
- $G = D_{2n}$, $|\Omega| = n$, $\Delta = \{1, 2\}$. $b(G) = 2$, $r(G) = \lceil \frac{n}{2} \rceil - 1$



P1 Determine $b(G)$.

P2 Classify G with $r(G) = 1$.

Trivial bounds $\log_{|\Omega|} |G| \leq b(G) \leq \log_2 |G|$.

Probabilistic method I (Liebeck & Shalev, 1999).

$$Q(G,c) = \frac{|\{(\alpha_1, \dots, \alpha_c) \in \Omega^c : G_{\alpha_1} \cap \dots \cap G_{\alpha_c} \neq 1\}|}{|\Omega|^c} \underset{\substack{\uparrow \\ c = b(G)}}{=} 1 - \frac{r(G) \cdot |G|}{|\Omega|^c}.$$

Note $Q(G,c) < 1 \iff b(G) \leq c$.

Suppose $G_{\alpha_1} \cap \dots \cap G_{\alpha_c} \neq 1$. Then $\exists x \in G_{\alpha_1} \cap \dots \cap G_{\alpha_c}$ of prime order. Thus, $\alpha_1, \dots, \alpha_c \in \mathrm{fix}_\Omega(x)$ and

$$Q(G,c) \leq \sum_{\substack{x \in G \\ |x| \text{ prime}}} \left(\frac{|\mathrm{fix}_\Omega(x)|}{|\Omega|} \right)^c$$

$$= \sum_{\substack{x \in G \\ |x| \text{ prime}}} \left(\frac{|x^G \cap G_\alpha|}{|x^G|} \right)^c =: \widehat{Q}(G,c)$$

Note $\widehat{Q}(G,c) < 1 \Rightarrow b(G) \leq c$

Primitive group $G_\alpha \underset{\max}{<} G$

(Equivalently, $\nexists$ nontrivial G-invariant partition of Ω).

e.g. $G = D_{2n}$, $|\Omega| = n$. Then G is primitive $\iff$ n is prime

- Bochert, 1889: $|\Omega| = n$. $G \neq A_n, S_n \Rightarrow b(G) \leq \frac{n}{2}$.

...

- Halasi, Liebeck & Maróti, 2019: $b(G) \leq 2 \log_{|\Omega|} |G| + 24$

Let T be a non-abelian simple group.

Holomorph $\mathrm{Hol}(T) = T : \mathrm{Aut}(T) \leq \mathrm{Sym}(T)$ primitive

- $b(\mathrm{Hol}(T)) = 3$.

Let $D = \{(t, \dots, t) : t \in T\} \leq T^k$

Then $T^k \leq \mathrm{Sym}(\Omega)$ with $\Omega = [T^k : D]$.

Note $G := N_{\mathrm{Sym}(\Omega)}(T^k) \cong T^k.(\mathrm{Out}(T) \times S_k)$ is a "diagonal type" primitive group on Ω.

Fawcett, 2013: $b(G) = 2$ only if $3 \leq k \leq |T| - 1$

Lemma $b(G) = 2 \iff \exists\, S \subseteq T$ s.t. $|S| = k$ & $\mathrm{Hol}(T)_{\{S\}} = 1$

Probabilistic method II (H, 2024).

Let $\mathrm{fix}(\sigma, k) = \{S \subseteq T : |S| = k \text{ \& } \sigma \in \mathrm{Hol}(T)_{\{S\}}\}$

Then $b(G) = 2$ if

$$\sum_{\substack{\sigma \in \mathrm{Hol}(T) \\ |\sigma| \text{ prime}}} |\mathrm{fix}(\sigma, k)| < \binom{|T|}{k}$$

Theorem (H, 2024)

If $3 \leq k \leq |T| - 3$ then $\exists\, S \subseteq T$ s.t. $|S| = k$ & $\mathrm{Hol}(T)_{\{S\}} = 1$

Theorem (Fawcett, 2013; H, 2024; FHLR, 2024+)

P1 and P2 are done if G is a "diagonal type" primitive group

§2 Rank

Rank # G_α - orbits on Ω (# G - orbits on Ω^2).

Example $G = GL_n(q)$, $\Omega = \mathbb{F}_q^n \setminus \{0\}$ $\Rightarrow$ rank$(G) = q$.

Note rank$(G) = 2$ $\Leftrightarrow$ G is 2-trans (classified via CFSG)

affine

almost simple

Burnside (~1900)

P3 Classify the rank 3 permutation groups.

Examples
- $G = GL_n(3)$, $\Omega = \mathbb{F}_3^n \setminus \{0\}$ - imprimitive
- $G = S_n$, $\Omega = \{$2-subsets of $[n]\}$ - primitive.

Primitive rank 3 groups: classified ~40yrs ago (Liebeck, Saxl, ...)

Lemma Let G be an imprimitive rank 3 group. Then
- $\exists!$ non-trivial G-invariant partition $\mathcal{B} = B^G$ of Ω.
- $G^{\mathcal{B}}$ and G_B^B are 2-transitive.

e.g. $G = GL_n(3)$, $\Omega = \mathbb{F}_3^n \setminus \{0\}$ $\Rightarrow$ $\mathcal{B} = \{$1-spaces$\}$. $G^{\mathcal{B}} = PGL_n(3)$, $G_B^B = S_2$