Permutation groups: bases and Saxl graphs.

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1. Bases

Let $G \leq \mathrm{Sym}(\Omega)$, where $|\Omega| < \infty$ and G is <u>transitive</u>.

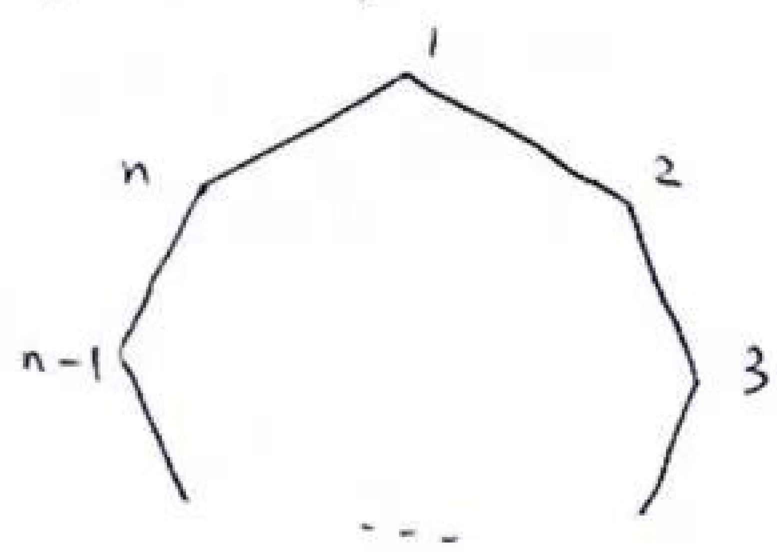
- Point stabiliser: $G_\alpha = \{ g \in G : \alpha^g = \alpha \}$.

<u>Note</u> $\bigcap_{\alpha \in \Omega} G_\alpha = 1$.

<u>Question</u> Any subset $\Delta \subseteq \Omega$ with $\bigcap_{\alpha \in \Delta} G_\alpha = 1$?

<u>Examples</u>

- $G = S_n$, $|\Omega| = n$, $\Delta = \{1, \dots, n-1\}$ $\boxed{b(G) = n-1}$
- $G = D_{2n}$, $|\Omega| = n$, $\Delta = \{1, 2\}$ $\boxed{b(G)=2}$



- $G = GL(V)$, $\Omega = V \setminus \{0\}$

Δ contains a basis of V. $\boxed{b(G) = \dim V}$

- $G = S_n$, $\Omega = \{ k\text{-subsets of } [n] \}$, $2k \leq n$

$\Delta = \{ \{1, \dots, k\}, \{2, \dots, k+1\}, \dots \{n-k+1, \dots, n\} \}$

$b(G) =$ smallest ℓ s.t.

$$\sum_{\substack{\pi \vdash n \\ \pi = (1^{c_1}, \dots, n^{c_n})}} (-1)^{n - \sum c_i} \frac{n!}{\prod i^{c_i} c_i!} \left(\sum_{\substack{\eta \vdash k \\ \eta = (1^{b_1}, \dots, k^{b_k})}} \prod \binom{c_j}{b_j} \right)^{\ell} \neq 0$$

by Mecenero & Spiga, 04/08/23

same (?) result by del Valle & Roney-Dougal 08/08/23.

Def. · $\Delta \subseteq \Omega$ is called a base for G if $\bigcap_{\alpha \in \Delta} G_\alpha = 1$.

· The base size of G, denoted $b(G)$, is the min size of a base for G.

Q1 Determine $b(G)$?

Q2 Bounds on $b(G)$?

Q3 Classify G with $b(G) = 2$?

Lower bound

Let Δ be a base of size $b(G)$ and $x, y \in G$. Then

$$\alpha^x = \alpha^y,\ \forall \alpha \in \Delta \iff x^{-1}y \in \bigcap_{\alpha \in \Delta} G_\alpha \iff x = y$$

That is,

$$\text{elements of } G \xleftrightarrow{1-1} \text{images of } \Delta$$

We have $|G| \leq |\Omega|^{b(G)}$, so $b(G) \geq \log_{|\Omega|} |G|$.

Upper bound

Write $\Delta = \{\alpha_1, \dots, \alpha_{b(G)}\}$ and $G^{(k)} = \bigcap_{i=1}^{k} G_{\alpha_i}$. Then

$$G \gneq G^{(1)} \gneq G^{(2)} \gneq \cdots \gneq G^{(b(G))} = 1.$$

Hence, $|G| \geq 2^{b(G)}$, so $b(G) \leq \log_2 |G| = \log_2 |\Omega| \cdot \log_{|\Omega|} |G|$

2. Primitive groups

Bounds

- Bochert, 1889: $|\Omega| = n$, $G \neq A_n$ or $S_n \Rightarrow b(G) \leq \frac{n}{2}$

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- Duyan, Halasi & Maróti, 2018:
$$b(G) \leq c \log_{|\Omega|} |G|$$
for some absolute constant c. (Pyber's conjecture, 1993)
- Halasi, Liebeck & Maróti, 2019: $b(G) < 2 \log_{|\Omega|} |G| + 24$.

Probabilistic method (Liebeck & Shalev, 1999)

$$Q(G,c) = \frac{|\{(\alpha_1, \dots \alpha_c) \in \Omega^c : \bigcap G_{\alpha_i} \neq 1\}|}{|\Omega|^c}$$

is the probability that a random c-tuple is NOT a base.

Note $b(G) \leq c \iff Q(G,c) < 1$.

We have

$$Q(G,c) \leq \sum_{\substack{x \in G \\ |x| \text{ prime}}} \left(\frac{|x^G \cap G_\alpha|}{|x^G|} \right)^c =: \hat{Q}(G,c)$$

Note $\hat{Q}(G,c) < 1 \Rightarrow b(G) \leq c$.

O'Nan - Scott

Finite primitive groups are divided into 5 types:

- Affine
- Almost simple
- Diagonal type
- Product type
- Twisted wreath product.

3. Diagonal type

Let T be a non-abelian simple group and let

$$X = \{ (x \ldots x) : x \in T \} \leq T^k$$

Then $T^k \leq \mathrm{Sym}(\Omega)$, where $\Omega = [T^k : X]$

A group G is said to be <u>diagonal type</u> if

$$T^k \trianglelefteq G \leq N_{\mathrm{Sym}(\Omega)}(T^k) \cong T^k . (\mathrm{Out}(T) \times S_k)$$

<u>Note</u> G induces $P_G \leq S_k$.

<u>Lemma</u> G is primitive $\iff$ P_G is primitive, or $\boxed{k=2 \text{ and } P_G = 1}$

$$T : \mathrm{Inn}(T) \trianglelefteq G \leq T : \mathrm{Aut}(T) = \mathrm{Hol}(T)$$

<u>Theorem</u> (Fawcett, 2013) $P_G \notin \{A_k, S_k\} \Rightarrow b(G) = 2$.

<u>Key observation</u>

$b(G) = 2$ if $\exists\, S \subseteq T$ s.t. $|S| = k$ & $\mathrm{Hol}(T)_{\{S\}} = 1$.

<u>Theorem</u> (H, 2023+) If $3 \leq k \leq |T| - 3$, then $\exists\, S \subseteq T$ s.t.

$|S| = k$ & $\mathrm{Hol}(T)_{\{S\}} = 1$.

<u>Theorem</u> (H, 2023+) $b(G) = 2 \iff$ one of the following holds:

(i) $P_G \notin \{A_k, S_k\}$

(ii) $3 \leq k \leq |T| - 3$

(iii) $k \in \{|T| - 2, |T| - 1\}$, and $S_k \ntrianglelefteq G$

<u>Theorem</u> (H, 2023+) Base sizes of diagonal type primitive groups are determined.

4. Saxl graph

Def (Burness & Giudici, 2020)

Let $G \leq \mathrm{Sym}(\Omega)$. Then the Saxl graph $\Sigma(G)$ is a graph with

- $V\Sigma(G) = \Omega$
- $\alpha \sim \beta \iff \{\alpha, \beta\}$ is a base for G.

$\boxed{\text{Now assume } b(G) = 2}$.

Example

- $G = PGL_2(q)$ and $\Omega = \{$ 2-subsets of $\{$ 1-spaces in $\mathbb{F}_q^2 \}\}$.

  Note $G_\alpha \cong D_{2(q-1)}$, and $\{\alpha, \beta\}$ is a base $\iff |\alpha \cap \beta| = 1$.

  Hence, $\Sigma(G) \cong \cancel{D}\ J(q+1, 2)$.

Note
- $\Sigma(G)$ is the union regular orbital graphs of G.
- G is primitive $\Rightarrow$ $\Sigma(G)$ is connected.

Conjecture (Burness & Giudici, 2020)

G primitive $\Rightarrow$ any two vertices of $\Sigma(G)$ have a common neighbour.

Example $G = PGL_2(q)$, $G_\alpha = D_{2(q-1)} \Rightarrow J(q+1, 2)$, satisfying BG conjecture.

Evidence:
- Chen & Du, 2023; Burness & H, 2022: $\mathrm{soc}(G) = PSL_2(q)$ ✓
- Burness & H, 2022: almost simple + G_α soluble ✓
- Lee & Popiel, 2023: some affine groups

Probabilistic method

Recall that $Q(G, 2) = 1 - \dfrac{\mathrm{val}(\Sigma(G))}{|\Omega|}$

Note · $Q(G, 2) < 1 \Leftrightarrow b(G) \leq 2 \Leftrightarrow E\Sigma(G) \neq \emptyset$

· $Q(G, 2) < \frac{1}{2} \Rightarrow \mathrm{val}(\Sigma(G)) > \frac{1}{2}|\Omega|$

$\Rightarrow \Sigma(G)$ has the common neighbour property.

Let $\Sigma(\alpha)$ be the set of neighbours of α in $\Sigma(G)$.

Recall BG conjecture: $\Sigma(\alpha)$ meets the union of regular G_β-orbits.

Conjecture (Burness & H, 2023)

G primitive, $\alpha, \beta \in \Omega \Rightarrow \Sigma(\alpha)$ meets every regular G_β-orbits.

Theorem (Burness & H, 2023)

BG conjecture $\Leftarrow$ BH conjecture.