Bases for permutation groups.

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1. Bases

Let $G \le \mathrm{Sym}(\Omega)$, where $|\Omega| < \infty$ and G is transitive

Point stabiliser: $G_\alpha = \{ g \in G : \alpha^g = \alpha \}$

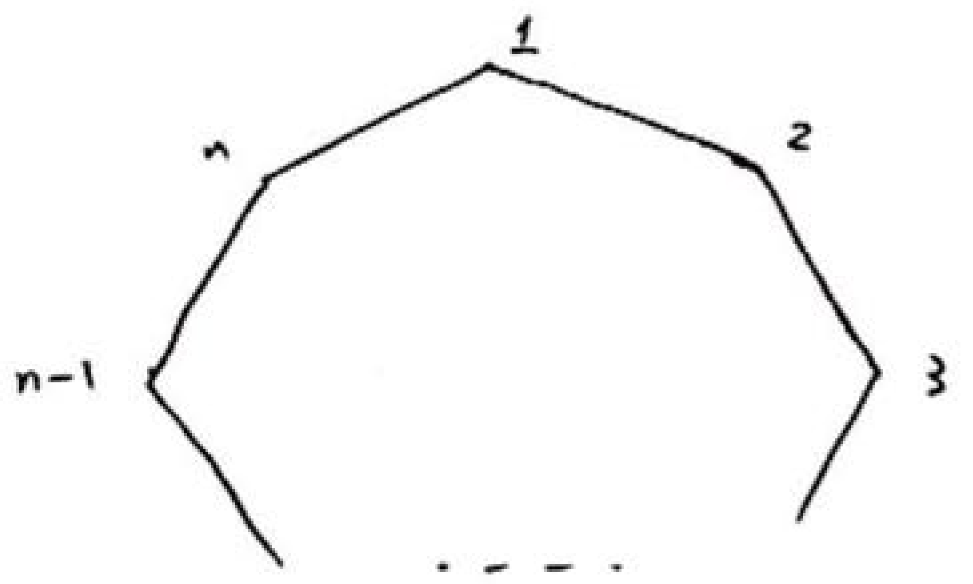
Note $\bigcap_{\alpha \in \Omega} G_\alpha = 1$.

Question Any subset $\Delta \subseteq \Omega$ with $\bigcap_{\alpha \in \Delta} G_\alpha = 1$?

Examples

- $G = S_n$, $|\Omega| = n$, $\Delta = \{1, \dots, n-1\}$ $\boxed{b(G) = n-1}$

- $G = D_{2n}$, $|\Omega| = n$, $\Delta = \{1, 2\}$ $\boxed{b(G) = 2}$



- $G = GL(V)$, $\Omega = V \setminus \{0\}$

  Δ contains a basis of V $\boxed{b(G) = \dim V}$

- $G = S_n$, $\Omega = \{ k\text{-subsets of } [n] \}$, $2k \le n$.

  $\Delta = \{ \{1, \dots, k\}, \{2, \dots, k+1\}, \dots \{n-k+1, \dots n\} \}$

$b(G) =$ smallest ℓ s.t.

$$\sum_{\substack{\pi \vdash n \\ \pi = (1^{c_1}, \dots, n^{c_n})}} (-1)^{n - \sum c_i} \frac{n!}{\prod i^{c_i} c_i!} \left(\sum_{\substack{\eta \vdash k \\ \eta = (1^{b_1}, \dots, k^{b_k})}} \prod \binom{c_j}{b_j} \right)^{\ell} \neq 0$$

by Meceneo & Spiga, 04/08/23

same (?) result by del Valle & Roney-Dougal 08/08/23

Def • $\Delta \subseteq \Omega$ is called a **base** for G if $\bigcap_{\alpha \in \Delta} G_\alpha = 1$.

• The base size of G, denoted $b(G)$, is the minimal size of a base for G.

Q1 Determine $b(G)$?

Q2 Bounds on $b(G)$?

Q3 Classify G with $b(G) = 2$?

Lower bound

Let Δ be a base of size $b(G)$ and $x, y \in G$. Then

$$\alpha^x = \alpha^y \ \forall \alpha \in \Delta \iff xy^{-1} \in \bigcap_{\alpha \in \Delta} G_\alpha$$
$$\iff x = y.$$

That is,

elements of G $\xleftrightarrow{1-1}$ images of Δ.

Hence, $|G| \leq |\Omega|^{b(G)}$ and so $b(G) \geq \log_{|\Omega|} |G|$.

Upper bound

Write $\Delta = \{\alpha_1, \ldots, \alpha_{b(G)}\}$ and $G^{(k)} = \bigcap_{i=1}^{k} G_{\alpha_i}$. Then

$$G \gneqq G^{(1)} \gneqq G^{(2)} \gneqq \cdots \gneqq G^{(b(G))} = 1.$$

Thus, $|G| \geq 2^{b(G)}$, so $b(G) \leq \log_2 |G|$.

Primitive groups

" Primitive " = " transitive " + " $G_\alpha <_{\max} G$ ".

e.g. $G = D_{2n}$, $|\Omega| = n$. Then G is primitive $\iff$ n is pri[me]

Halasi, Liebeck & Maróti, 2019: $b(G) \leq 2 \log_{|\Omega|} |G| + 24$.

(originally Pyber's conjecture).

O'Nan - Scott

Finite primitive groups are divided into 5 types:

- Affine
- Almost simple
- Diagonal type
- Product type
- Twisted wreath product

2. Diagonal type

Let T be a non-abelian finite simple group and let
$$D = \{ (t, \cdots, t) : t \in T \} \leq T^k.$$

Then $T^k \leq Sym(\Omega)$ with $\Omega = [T^k : D]$.

A group G is said to be of <u>diagonal type</u> if
$$T^k \trianglelefteq G \leq N_{Sym(\Omega)}(T^k) \cong T^k . (Out(T) \times S_k)$$

<u>Note</u> G induces $P_G \leq S_k$, so $T^k \trianglelefteq G \leq T^k . (Out(T) \times P_G)$.

<u>Lemma</u> G is primitive $\Longleftrightarrow$ P_G is primitive, or $k = 2$ and $P_G = 1$

$$T : Inn(T) \trianglelefteq G \leq T : Aut(T) = Hol(T).$$

<u>Theorem</u> (Fawcett, 2013)

- $P_G \notin \{A_k, S_k\} \Rightarrow b(G) = 2$
- $P_G \in \{A_k, S_k\}$ and $b(G) = 2 \Rightarrow 2 < k < |T|$.

<u>Key observation</u>.

$b(G) = 2$ if $\exists\, S \subseteq T$ s.t. $|S| = k$ and $Hol(T)_{\{S\}} = 1$.

An approach.

Let $\mathcal{A} = \{S \subseteq T : |S| = k \text{ and } \mathrm{Hol}(T)_{\{S\}} \neq 1\}$.

Suppose $S \in \mathcal{A}$.

Then $\exists\, \sigma \in \mathrm{Hol}(T)_{\{S\}}$ of prime order.

Thus,

$$S \in \mathrm{fix}(\sigma, k) = \{S \subseteq T : |S| = k \text{ and } \sigma \in \mathrm{Hol}(T)_{\{S\}}\}.$$

Let $\mathcal{P}$ be the set of elements of $\mathrm{Hol}(T)$ of prime order

Then

$$|\mathcal{A}| = \left| \bigcup_{\sigma \in \mathcal{P}} \mathrm{fix}(\sigma, k) \right|$$

$$\leq \sum_{\sigma \in \mathcal{P}} |\mathrm{fix}(\sigma, k)| =: m.$$

Note $b(G) = 2$ if $m < \binom{|T|}{k}$.

Main results.

Theorem (H, 2023+) If $3 \leq k \leq |T| - 3$, then $\exists\, S \subseteq T$ s.t. $|S| = k$ and $\mathrm{Hol}(T)_{\{S\}} = 1$.

Theorem (H, 2023+) $b(G) = 2$ iff:

- $P_G \notin \{A_k, S_k\}$
- $3 \leq k \leq |T| - 3$
- $k \in \{|T| - 2, |T| - 1\}$ and $S_k \nleq G$.

Theorem (H, 2023+) The precise base size of every primitive group of diagonal type is determined.

3. Regular orbits

Note G has a regular orbit on $\Omega^k \iff b(G) \le k$.

Let $r(G)$ be the number of regular orbits on $\Omega^{b(G)}$.

Problem Classify the transitive groups G with $r(G) = 1$?

Burness & H, 22/23 : G almost simple primitive, G_α soluble ✓

e.g. $G = PGL_2(q)$, $G_\alpha = D_{2(q-1)}$

(note that $\Omega = \{$ 2-subsets of $\{$1-spaces of $\mathbb{F}_q^2\}\}$)

H, 2023+ : G diagonal type, $b(G) = 2$ ✓

$G = T^k.(\mathrm{Out}(T) \times S_k)$, $T = A_5$, $k \in \{3, 57\}$.

Freedman, H, Lee & Rekvényi, 2023+ :

G diagonal type, $b(G) > 2 \Rightarrow r(G) > 1$.