# Bases and Saxl graphs for permutation groups.

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Throughout, let $G \leq \mathrm{Sym}(\Omega)$ and assume $|\Omega| < \infty$.

## §1 Bases.

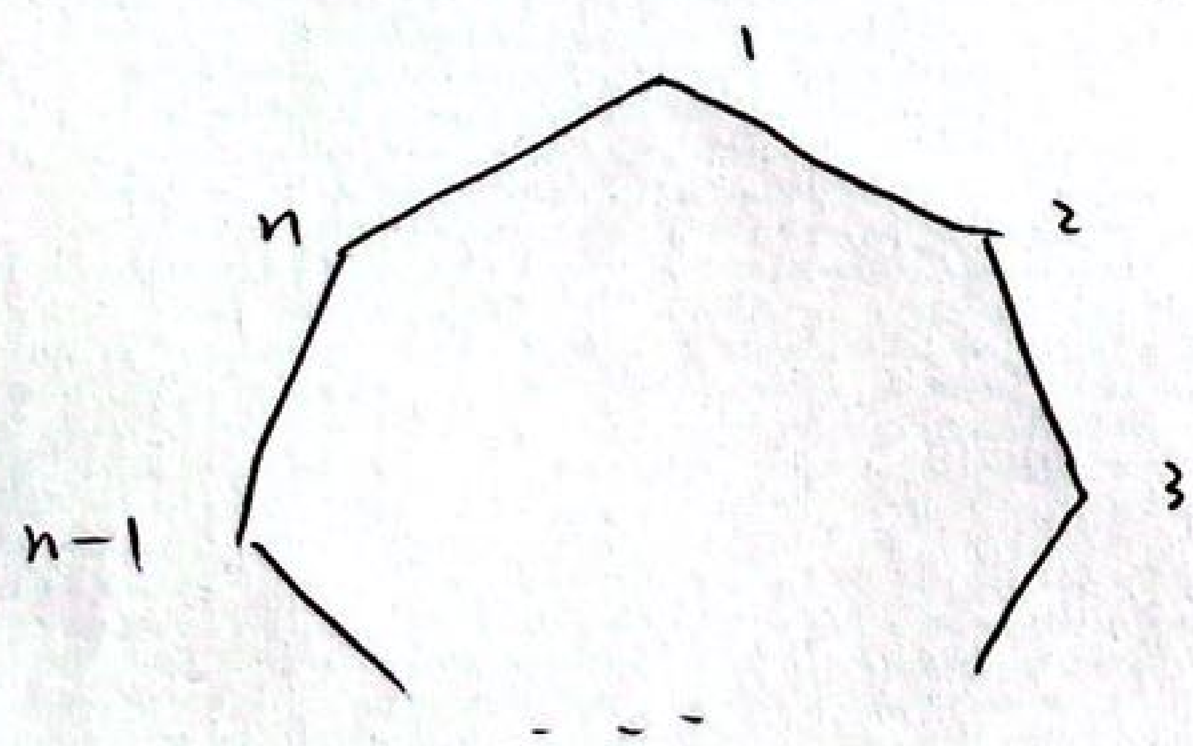
<u>Bases</u> $\Delta \subseteq \Omega$ s.t. $\bigcap_{\alpha \in \Delta} G_\alpha = 1$.

<u>Base size $b(G)$</u> minimal size of a base for $G$.

<u>Examples</u>

- $G = S_n$, $|\Omega| = n$, $\Delta = \{1, \dots, n-1\}$, $b(G) = n-1$.
- $G = GL(V)$, $\Omega = V \setminus \{0\}$.

  $\Delta$ contains a basis of $V$. $b(G) = \dim V$.
- $G = D_{2n}$, $|\Omega| = n$, $\Delta = \{1, 2\}$. $b(G) = 2$




- $T$ non-abelian simple, $\Omega = T$, $G = T : \mathrm{Aut}(T) = \mathrm{Hol}(T)$

  $G_1 = \mathrm{Aut}(T)$ ; $G_1 \cap G_x = C_{\mathrm{Aut}(T)}(x) \neq 1 \Rightarrow b(G) \geq 3$.

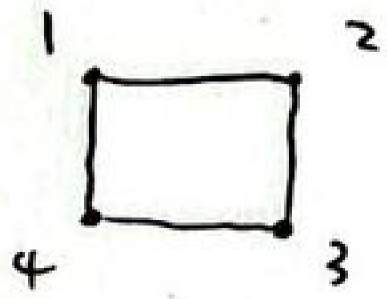
<u>Fact</u> $\exists x, y \in T$ s.t. $\langle x, y \rangle = T \Rightarrow b(G) = 3$.

## §2 Saxl graphs

**Saxl graph $\Sigma(G)$** (Burness & Giudici, 2020)

vertex set $\Omega$; $\alpha \sim \beta \iff \{\alpha, \beta\}$ is a base.

**Examples**

- $G = D_8$, $\Omega = \{1,2,3,4\}$: $\Sigma(G) =$ 

$\cong K_{2,2}$.
- $G = D_8^k$, $\Omega = \{1,2,3,4\}^k$:

  $(\alpha_1, \dots, \alpha_k) \sim (\beta_1, \dots, \beta_k) \iff$ each $\{\alpha_i, \beta_i\}$ is a ba

  So $\Sigma(G) \cong 2^{k-1} K_{2^k, 2^k}$.
- $G = PGL_2(q)$, $\Omega = \{$2-subsets of $\{$1-spaces of $\mathbb{F}_q^2\}\}$

  Then $G_\alpha \cong D_{2(q-1)}$, and $\alpha \sim \beta \iff |\alpha \cap \beta| = 1$

  So $\Sigma(G) \cong J(q+1, 2)$ is a Johnson graph.

**Generalised Saxl graph $\Sigma(G)$** (Freedman, H, Lee & Rekvényi, 2024

vertex set $\Omega$; $\alpha \sim \beta \iff \{\alpha, \beta\}$ is a subset of a base of size $b(G$

**Examples**

- $G = S_n$, $|\Omega| = n$ : $\Sigma(G)$ is complete
- $G$ is 2-transitive : $\Sigma(G)$ is complete
- $G = GL(V)$, $\Omega = V \setminus \{0\}$ : $\Sigma(G)$ is complete multipartite
- $G = \mathrm{Hol}(T)$, $\Omega = T$ : $\Sigma(G)$ is complete

  Fact $\forall\, 1 \neq x \in T$, $\exists\, y \in T$ s.t. $\langle x, y \rangle = T$.

**Basic properties**

- $G \leqslant \mathrm{Aut}(\Sigma(G))$
- $G$ is transitive $\Rightarrow$ $\Sigma(G)$ is $G$-vertex-transitive
- $G_1 \leqslant \mathrm{Sym}(\Omega_1)$, $G_2 \leqslant \mathrm{Sym}(\Omega_2)$, $b(G_1) = b(G_2)$

  $\Rightarrow \Sigma(G_1 \times G_2) = \Sigma(G_1) \times \Sigma(G_2)$ (on $\Omega_1 \times \Omega_2$).

## Probability

Let $Q(G,k) = \dfrac{|\{(\alpha_1,\dots,\alpha_k) \in \Omega^k : G_{\alpha_1} \cap \dots \cap G_{\alpha_k} \neq 1\}|}{|\Omega|^k}$

Note $Q(G,k) < 1 \iff b(G) \leq k$.

Lemma If $t \in \mathbb{N}$, $t \geq 2$ and $Q(G, b(G)) < 1/t$, then

- Any $t$ vertices in $\Sigma(G)$ have a common neighbour.
- $\Sigma(G)$ is connected with diameter $\leq 2$.
- $\Sigma(G)$ has clique number $\geq t+1$
- $\Sigma(G)$ is Hamiltonian.

## Connectedness

Note If $C$ is a connected component of $\Sigma(G)$, then

$$C^g \cap C = C \text{ or } \emptyset \quad \forall g \in G.$$

Thus, $G_\alpha < G_{\{C\}}$ if $\alpha \in C$.

Primitive group $G_\alpha \underset{\max}{<} G$.

- $G$ is primitive $\Rightarrow$ $\Sigma(G)$ is connected

## §3 Problems & results.

1. $b(G)$ for primitive groups

O'Nan-Scott : Finite primitive groups are divided into 5 types

- Almost simple: $T \trianglelefteq G \leq \mathrm{Aut}(T)$, $T$ simple
    - Precise $b(G)$ when $T = A_n$ or sporadic. ✓
    - Precise $b(G)$ when $G_\alpha$ soluble (Burness, 2021) ✓
- Diagonal type (e.g. $\mathrm{Hol}(T)$)
    - Precise $b(G)$ in every case (Fawcett, 2013; H, 2024)
- Other types : Partial results.

2. Common neighbour

Conjecture (BG, 2020 ; FHLR, 2024 +)

$G$ primitive $\Rightarrow$ any two vertices of $\Sigma(G)$ have a common neighbour

Recall $Q(G, b(G)) < \frac{1}{2}$ is sufficient.

Evidence

- $PSL_2(q) \trianglelefteq G \leq P\Gamma L_2(q)$ (BH, 2022 ; FHLR, 2024 +)
- $G$ almost simple, $G_\alpha$ soluble (BH, 2022 ; FHLR, 2024 +).
- $G$ almost simple sporadic & $b(G) \geq 3$ (FHLR, 2024+)
- Some diagonal type groups (H, 2024 +)

3. Arc - transitivity

Let $\mathrm{reg}(G)$ be the number of regular $G$-orbits on $\Omega^{b(G)}$

Equivalently, $\mathrm{reg}(G) = \dfrac{|\Omega^k|(1 - Q(G, b(G)))}{|G|}$

Note $\mathrm{reg}(G) = 1 \Rightarrow \Sigma(G)$ is $G$-arc-transitive.

Problem Classify the primitive groups $G$ with $\mathrm{reg}(G) = 1$.

- $G$ almost simple, $G_\alpha$ soluble ✓ (BH, 2022/23)

  e.g. $(G, G_\alpha) = (PGL_2(q), D_{2(q+1)})$

- $PSL_2(q) \trianglelefteq G \leq P\Gamma L_2(q)$ ✓ (FHLR, 2024 +)
- $G$ diagonal type ✓ (H, 2024 ; FHLR, 2024 +)

4. Completeness

Note If $b(G) = 2$, then $\Sigma(G)$ is complete $\Leftrightarrow$ $G$ is Frobenius

Problem Classify the primitive groups $G$ s.t. $\Sigma(G)$ is complete

e.g. 2-transitive groups ; $\mathrm{Hol}(T)$

FHLR, 2024 + : $PSL_2(q) \trianglelefteq G \leq P\Gamma L_2(q)$ ✓.