Bases for permutation groups

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Remark. The **determining number** of a graph Γ is $b(\mathrm{Aut}(\Gamma))$.

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$b(G)$ = smallest integer ℓ such that

$$\sum_{\substack{\pi \,\vdash\, n \\ \pi=(1^{c_1},\ldots,n^{c_n})}} (-1)^{n-\sum_{i=1}^n c_i} \frac{n!}{\prod_{i=1}^n i^{c_i} c_i!} \left(\sum_{\substack{\eta \,\vdash\, k \\ \eta=(1^{b_1},\ldots,k^{b_k})}} \prod_{j=1}^k \binom{c_j}{b_j} \right)^{\ell} \neq 0.$$

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Problem. Determine the base sizes of primitive groups.

O'Nan-Scott theorem

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- Diagonal type **???**
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Note. $b(G) = 2$ if $m < \binom{|T|}{k}$.

Main results

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Suppose $G \leqslant T^k.(\mathrm{Out}(T) \times P)$ is a diagonal type primitive group with top group P. Then $b(G) = 2$ if and only if one of the following holds:

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Theorem (H, 2023+)

Suppose G is a diagonal type primitive group. Then $b(G)$ is computed. ✓

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Freedman, H, Lee & Rekvényi, 2023+:

$$G \text{ is a diagonal type primitive group with } b(G) > 2 \implies r(G) > 1.$$

Thank you!