Base-two primitive groups and their Saxl graphs

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Bases

Let G be a transitive permutation group on a finite set Ω with point stabiliser H. A **base** for G is a subset $\Delta \subseteq \Omega$ such that

$$\bigcap_{\alpha\in\Delta} G_\alpha = 1.$$

The minimal size of a base for G is called the **base size** of G and denoted $b(G)$.

Example. Assume $G = \mathrm{GL}(V)$ and $\Omega = V \setminus \{0\}$. Then $\Delta \subseteq V$ is a base for G if and only if Δ contains a basis of V. In particular, $b(G) = \dim V$.

An ambitious project initiated by Jan Saxl in the 1990s is:

> Classify the finite primitive groups G with $b(G) = 2$.

See [2, Section 1] for a brief summary of progress towards this goal.

Saxl graphs

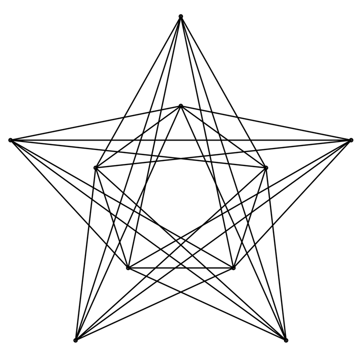
To study base-two groups, Burness and Giudici [1] introduced the following graph.

Definition. The **Saxl graph** $\Sigma(G)$ of G is a graph with vertices Ω and two vertices are adjacent if they form a base.

Example. Assume $G = \mathrm{PGL}_2(q)$ and

$$\Omega = \{\text{distinct pairs of } 1\text{-subspaces of } \mathbb{F}_q^2\}.$$

Then two elements in Ω form a base if and only if they are not disjoint. It follows that $\Sigma(G)$ is isomorphic to the **Johnson graph** $J(q+1,2)$. In particular, if we take $q = 4$ then $G \cong A_5$, $H \cong S_3$ and the Saxl graph $\Sigma(G)$ coincides with the complement of the Petersen graph:



We record some basic properties of Saxl graphs of base-two transitive groups.

- $\Sigma(G)$ is G-vertex-transitive and is the union of regular **orbital graphs** of G;
- The valency of $\Sigma(G)$ is $v(G) = r(G)|H|$, where $r(G)$ is the number of regular H-orbits on Ω;
- If G is primitive, then $\Sigma(G)$ is connected.

One of the main problems in the study of Saxl graphs is:

> **Conjecture** (Burness-Giudici)**.** Let G be a base-two primitive permutation group. Then any two vertices in $\Sigma(G)$ have a common neighbour.

In particular, $\Sigma(G)$ has diameter at most 2. In the above example, G is base-two primitive and $\Sigma(G) \cong J(q+1,2)$ has the common neighbour property.

Main results

We list some evidence for the Burness-Giudici conjecture.

- All primitive groups of degree up to 4095;
- "Most" almost simple primitive groups with alternating or sporadic socle;
- $\mathrm{soc}(G) = \mathrm{PSL}_2(q)$ (see [3, Theorem 4.22]).

Let $\mathcal{B}$ be the set of almost simple base-two primitive groups with soluble point stabilisers. The following theorem is proved in [3].

> **Theorem** (Burness & H)**.** If $G \in \mathcal{B}$ then any two vertices in $\Sigma(G)$ have a common neighbour.

It is proved in [2] that the Burness-Giudici conjecture is equivalent to the following "stronger" statement. Here $\Sigma(\alpha)$ is the set of neighbours of α in $\Sigma(G)$.

> **Conjecture.** Let $G \leqslant \mathrm{Sym}(\Omega)$ be a base-two primitive permutation group and $\alpha, \beta \in \Omega$. Then $\Sigma(\alpha)$ meets every regular G_β-orbit.

See [2, Section 5] for some evidence of this conjecture.

Related results

Some results on the valency of Saxl graphs are presented in [4], including a general method for computing $r(G)$.

Note that $r(G) \geqslant 1$ if and only if $b(G) \leqslant 2$. The following problem concerns the extremal case.

> Classify the finite primitive groups G with $r(G) = 1$.

This is the case where $\Sigma(G)$ is an orbital graph of G. In [3, Theorem 4] we classified the groups $G \in \mathcal{B}$ with $r(G) = 1$ (e.g. $(G,H) = (\mathrm{PGL}_2(q), D_{2(q-1)})$).

In [3], we also studied the **clique number** $\omega(G)$ and the **independence number** $\alpha(G)$ of $\Sigma(G)$.

> **Theorem** (Burness & H)**.** If $G \in \mathcal{B}$ is simple, then either $(G,H) = (A_5, S_3)$, or $\omega(G) \geqslant 5$ and $\alpha(G) \geqslant 4$.

Probabilistic methods

This method was first introduced by Liebeck and Shalev [5]. Consider

$$Q(G) := \frac{|\{(\alpha,\beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{|\Omega|^2} = 1 - \frac{v(G)}{|\Omega|},$$

the probability that a random pair in Ω is not a base.

Note that $b(G) \leqslant 2$ if and only if $Q(G) < 1$. Intuitively, if $Q(G)$ is small then $\Sigma(G)$ contains many edges. More specifically, we have the following.

- If $Q(G) < 1/2$, then any two vertices in $\Sigma(G)$ have a common neighbour.
- If $Q(G) < 1/t$ for some integer $t \geqslant 2$, then $\omega(G) \geqslant t+1$.

In general, it is difficult to compute $Q(G)$ precisely, but

$$Q(G) \leqslant \sum_{x\in\mathcal{P}} \frac{|x^G \cap H|}{|x^G|} =: \widehat{Q}(G),$$

where $\mathcal{P}$ is the set of elements of prime order in G. In particular, $b(G) \leqslant 2$ if $\widehat{Q}(G) < 1$, and we can use $\widehat{Q}(G)$ to obtain a lower bound on $r(G)$.

References

[1] T.C. Burness and M. Giudici, *On the Saxl graph of a permutation group*, Math. Proc. Cambridge Philos. Soc. **168** (2020), 219–248.

[2] T.C. Burness and H.Y. Huang, *On base sizes for primitive groups of product type*, submitted (2022), arXiv:2202.02816.

[3] T.C. Burness and H.Y. Huang, *On the Saxl graphs of primitive groups with soluble stabilisers*, Algebr. Comb., to appear.

[4] J. Chen and H.Y. Huang, *On valency problems of Saxl graphs*, J. Group Theory **25** (2022), 543–577.

[5] M.W. Liebeck and A. Shalev, *Simple groups, permutation groups, and probability*, J. Amer. Math. Soc. **12** (1999), 497–520.