Regular orbits of primitive groups on power sets

Hong Yi Huang

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University of BRISTOL

Outline

- O'Nan-Scott
- Some observations
- Minimal degrees
- Distinguishing numbers
- Applications (bases, 2-arc-transitive graphs)

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AS	G	G primitive on $\{1, \ldots, n\}$

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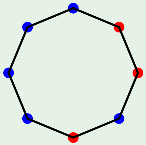

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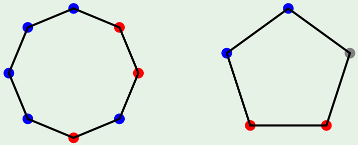

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Let G be primitive. Then one of the following holds.

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Theorem (Burness & H, 2022)

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This implies $|X| = 2^n - \binom{n}{n/2} \geqslant 2^{n-1}$.

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The lemma follows by observing that

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Indeed,

$$f(|T|, k) \geqslant f(|T|, 2) = \frac{2^{|T|/3-1}}{2|T|^3} > 1.$$

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Remark. By random search, $2^4{:}\mathrm{O}_4^-(2)$ has two regular orbits on $\mathcal{P}(\Omega)$.

Random search

Let $G = \mathrm{AGL}_8(2)$. Note that $n = 2^8$ is too large to use `Subset({1..n})`. We use the following code to obtain a regular G-orbit on X.

```
G:=AGL(8,2);
n:=Degree(G);
repeat
A:=[];
for i in [1..20] do
Append(~A,Random([1..n]));
end for;
A:={x :  x in A};
m:=#Stabilizer(G,A);
[#A,m];
until #A ne (n div 2) and m eq 1;
```

This returns `[19,1]` in 118.360 seconds.

Almost simple groups

Theorem (CFSG)

A non-abelian finite simple group is isomorphic to one of the following.

- Alternating group A_n, $n \geqslant 5$;
- Classical simple group:

  $\mathrm{L}_n^\epsilon(q)$, $\mathrm{PSp}_{2n}(q)$, $\mathrm{P}\Omega_{2n+1}(q)$, $\mathrm{P}\Omega_{2n}^\epsilon(q)$;
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Theorem (CFSG)

A non-abelian finite simple group is isomorphic to one of the following.

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A group G is almost simple if $\mathrm{soc}(G)$ is non-abelian simple.

G is almost simple iff $T \trianglelefteq G \leqslant \mathrm{Aut}(T)$ for some non-abelian simple T.

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- $\mathrm{soc}(G) = A_m$, G_α imprimitive $\implies \mu(G) \geqslant n/2$
- $\mathrm{soc}(G) = A_m$, G_α primitive $\implies \mu(G) \geqslant 2n/3$
- G classical $\implies \mu(G) \geqslant 3n/7$
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Lower bounds for n:

- $\mathrm{soc}(G) = A_m$, G_α primitive: $n > |G|/3^m$ **(Maróti, 2002)**
- G Lie type: **Guest, Morris, Praeger & Spiga, 2015**
- G sporadic: **Wilson, 2017, or Web Atlas**

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Remark. Using above bounds, only a few almost simple primitive groups survive the cut $|G| < 2^{\mu(G)/2-1}$, which can all be handled by random search.

Distinguishing numbers

Let

$$\mathcal{P}_m(\Omega) = \{(\pi_1, \ldots, \pi_m) \,:\, \pi_i \subseteq \Omega,\ \pi_i \cap \pi_j = \emptyset \text{ for } i \neq j,\ \bigcup_i \pi_i = \Omega\}$$

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- $G \leqslant T^k.(\mathrm{Out}(T) \times P)$ diagonal type, $P \neq A_k, S_k$ primitive: $r(G) > 1$ **(Fawcett, 2013; H, in progress)**

Base-two diagonal type group

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Remark. Indeed, $P \neq A_k, S_k \implies b(G) = 2$ **(Fawcett, 2013)**

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$$(G, |V\Gamma|) = (D_{10}, 5), (3^2{:}D_8, 9), (S_5, 10), (\mathrm{GL}_2(3), 8)$$

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Note. $(G, |V\Gamma|) = (\mathrm{GL}_2(3), 8) \implies \Gamma = K_{2,2,2,2}$ is not 2-arc-transitive.

Thank you!